Competition Policy & Game Theory

Chapitre I
Introduction to Game theory

Roadmap

0. Introduction
1. Games, Extensive and Strategic (normal) form
2. Solution concepts : Dominant strategies equilibrium, Nash equilibrium and Rationalizability
3. Existence and efficiency

Introduction

Non cooperative game and Competition policy

In economic theory, imperfect competition is a type of market structure showing some but not all features of competitive markets.

This course introduces game theory to serve the analysis of anti-competitive behavior (we will use the generic term collusion) and to enlighten its implications for competition policy.

We should notice that modern approach of the study of collusion is to use non-cooperative game theory, in which firms are supposed to maximise their own profit individually. The course will follow this mainstream.

In non cooperative games, we consider situations in which each firm maximises its own profit individually.

However, the issue of a game depends on the decision of many persons ; A naive approach could relate the context of each player-decision maker to uncertainty, while game theory will explore the idea of coordination.

The central issue of non cooperative game is that each player has always the freedom of making an unilateral deviation : there is no institution which could bound a player to commit on a particular strategy.

Game and interdependencies

The distinctive feature of a game is the presence of interdependencies among the agents : one agent's utility depends not only on his own actions, but also on the actions of each of the other agents.

It is the agents' awareness of interactions among their decision which give rise to the subtle problems of game theory.

In markets, independencies of the firms often come from the fact that many firms serve a pool of consumers.

Information of the players

The information that the players are endowed appears to be a key element for the analysis of a game.

We will introduce later perfect or imperfect information. Typically, whenever the agents does not have the same knowledge about the game, that could increase the uncertainty about the outcome. We will see that this idea is not necessary true.

One central question In Game theory is about the coordination and the fact that the agents will or not share the same understanding and the same beliefs about equilibrium.

1. Games, Extensive and Strategic (normal) form

What is a game

In a game, a finite (sometimes infinite) number of rational agents have to take simultaneous decisions, Which affects the welfare of every body. In game theory, a player's strategy is any of the options he or she can choose in a setting where the outcome depends not only on their own actions but on the action of others. Then, action the player will take, at any stage of the game, Will depend upon its beliefs on the the other players ' strategy.

A game is described by

- the list of players
- the rules of the games, the allowed actions of the players and the interactions
- the payoff resulting in any realizable history

Exemple

(VAYDAY) or gladiators ' game

Two identical players, at each side of a cord must simultaneously choose an effort level ; the one which effort is greater is the winner. In case there is no winner, they are *ex aequo*. The effort variable $e \in [0, 1]$. We let e_a and e_b denote the respective effort of each of the players.
Here are the payoff functions :

- for the winner : $1 - e$
- in case of ex eaquo : $\max(\frac{1}{2} - e, 0)$
- for the looser : 0

- Find the Unique equilibrium. Interpret the result, considering that 0 could signify death.

Exemple 2 : a simple model of competition between firms

Let consider the following competition game between two firms, A and B. Both of them share a market in which there is a continuum of agents. Each buyer reservation price is equal to 1. Each firm cost is equal to zero. The game is simultaneous : whenever $1 \geq p_A$ and $p_A < p_B$, firm A wins all the market, $q_A = 1$ whenever $1 \geq p_A = p_B$, there is a tie break rule : the market is divided among the competitors and $q_A = 1/2$. Firm i 's payoff is :

$$\pi_i = q_i p_i$$

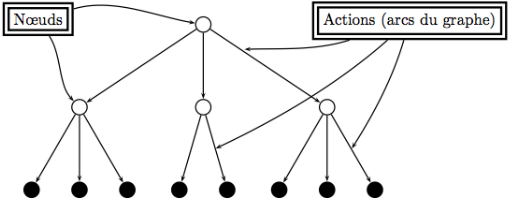
What is the (non cooperative) equilibrium of this game ?

What is the cooperative issue of this game ?

Is the introduction of time relevant in that game ?

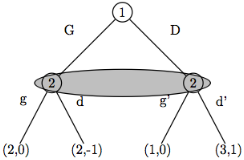
Extensive form game

An extensive-form game is a specification of a game in game theory, allowing (as the name suggests) for the explicit representation of a number of key aspects, like the sequencing of players' possible moves, their choices at every decision point, the (possibly imperfect) information each player has about the other player's moves when they make a decision, and their payoffs for all possible game outcomes. In a first approach, we first consider the extensive-form game as being just a game tree with payoffs. Each action is described by a branch ; Each branch starts from a node that is managed by one particular player. A tree starts with an initial node and every node is linked to the initial node by one unique way, composed of branchs and nodes. Payoffs are associated to every terminal node.



Information and extensive form game

We say that there is perfect information under two conditions (I) All players know the game structure (including the payoff functions at every outcome). (ii) Each player, when making any decision, is perfectly informed of all the events that have previously occurred.



The idea of information set is at the root of imperfect information. An *information set* is a set of decision nodes such that :

- Every node in the set belongs to one player.
- When the game reaches the information set, the player who is about to move cannot differentiate between nodes within the information set ; i.e. if the information set contains more than one node, the player to whom that set belongs does not know which node in the set has been reached.
- The set of realizable actions is then the same at each node belonging to an information set

An exemple of extensive form game

Let consider a non divisible good, two sellers, 1 and 2, one buyer. The buyer buys at least one good in each period. Her reservation price is 1.

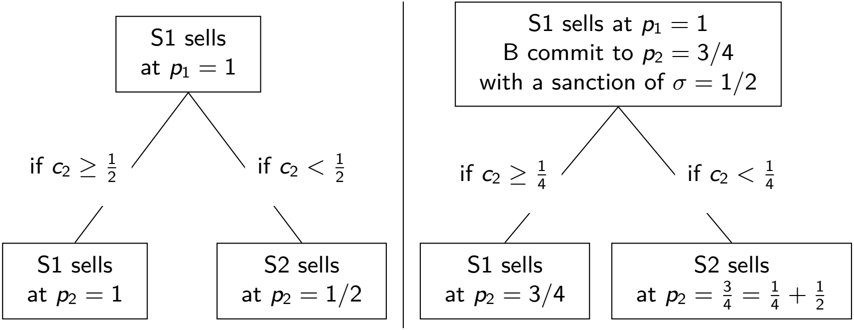
Seller 1 's is alone in period one, and its cost is $c_1 = 1/2$, constant for each period. Seller 2 is not on the market at Period 1, and its cost uniformly distributed in $[0, 1]$ is only determined at period 2.

A simple and strong statement : the good should be sold at period 1 and at period 2 by the seller which cost is the lowest.The good is always sold because the reservation price of the seller is greater than the cost, whatever it will be.

The question : What could do S1 to prevent the entry of S2 ?

Aghion Bolton (1987) continued

There are many scenarios possible. Let consider in particular two of them :



We will not enter in the details, but you should keep in mind that dynamics offers tremendous opportunities to firms.

Normal form game

In game theory, normal form is a description of a game. Unlike extensive form, normal-form representations are not graphical per se, but rather represent the game by way of a matrix. The normal-form representation of a game includes all perceptible and conceivable strategies, and their corresponding payoffs, for each player.

Each agent, i chooses a strategy a_I from the set 1_I of possible strategies. Because it is a game, each agents's utility, u^I depends on every agents' strategy : $u^i = u^i(a_1, \ldots, a_n)$, $i = 1, \ldots, n$.

A strategy is a complete description of the agent's planned action.

Normal form and extensive form

A strategy space for a player is the set of all strategies available to that player, whereas a strategy is a complete plan of action for every stage of the game, regardless of whether that stage actually arises in play. A payoff function for a player is a mapping from the cross-product of players' strategy spaces to that player's set of payoffs

While this approach can be of greater use in identifying strictly dominated strategies and Nash equilibria, some information is lost as compared to extensive-form representations. The normal-form representation of a game includes all perceptible and conceivable strategies, and their corresponding payoffs, for each player.

However, one important step of the analysis of an extensive form is to look for the strategies that the player will form before the beginning of the game. So that an extensive game can always be represented as a normal form game.

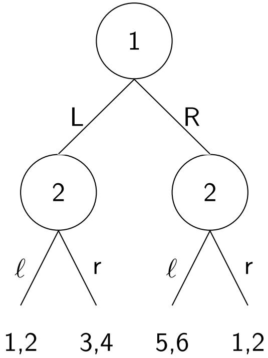
Normal and extensive form, simultaneous and dynamic games

Framing : simultaneous and dynamic games In economics, the framing, the way the history is told matter. Simultaneous game present situations in which different players have to choose *simultaneously* an action ; such a game is naturally represented by a normal form game. In a dynamic game, there is a specific *timing* that describes precisely the date at which each player will play. Such a game is naturally represented as an extensive form game.

Representation : Normal and extensive form games Every game have two ways to represent it : an extensive form game and a normal form game. A simultaneous game can be represented by a tree, in which, at each date, the player doesn't know the node at which she should make her decision. A dynamic game, is characterized by each player strategy, and then, there is a simultaneous strategy choices, made before the start of the game, which is a normal form representation of the game.

Normal form and extensive form

Consider the following extensive game with two players, 1 and 2 :



2 Strategies of player 1 : $\{L, R\}$, 4 strategies of player 2 : $\{ll, lr, rl, rr\}$

one equilibrium, by backward induction : $(R,\ rl)$. Are there other equilibria ?

To answer to the question addressed to the preceding game, may be it could be more useful putting it in normal form

	ll	lr	rl	rr
L	1,2	1,2	3,4	3,4
R	5,6	1,2	5,6	1,2

Check if (R,ll) (L,rr) and (L,lr) could also be or not an equilibrium

2. Solution concept, particularly, Dominant strategies equilibrium, Nash equilibrium and Rationalizability.

Solution concept

In game theory, a solution concept is a formal rule for predicting how a game will be played. These predictions are called "solutions", and describe which strategies will be adopted by players and, therefore, the result of the game. The most commonly used solution concepts are equilibrium concepts, most famously Nash equilibrium.

Many solution concepts, for many games, will result in more than one solution. This puts any one of the solutions in doubt, so a game theorist may apply a refinement to narrow down the solutions. Each successive solution concept presented in the following improves on its predecessor by eliminating implausible equilibria in richer games.

Dominant equilibrium

Definition : a player 's strategy is dominant, when, it is not possible to increase the player's payoff by the choice of another strategy, regardless of any other agent's strategy.

Notice that in most of the game, there is no dominant strategy.

Definition

Whenever every players have a dominant strategy, the collection of all those strategies form a dominant equilibrium, each agent's best action is uniquely defined regardless of any other agent's action.

Prisoner's dilemma

Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the pair on the principal charge, but they have enough to convict both on a lesser charge. Simultaneously, the prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent. The offer is :

- If A and B each betray the other, each of them serves two years in prison
- If A betrays B but B remains silent, A will be set free and B will serve three years in prison (and vice versa)
- If A and B both remain silent, both of them will only serve one year in prison (on the lesser charge).

	S	B
s	-1,-1	-3,0
b	0,-3	-2,-2

Nash equilibrium

Most games do not have a dominant equilibrium. A less stringent notion of equilibrium is required.

Definition

If each player has chosen a strategy and no player can benefit by changing strategies while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitutes a Nash equilibrium.

At a Nash equilibrium, each agent does the best he can given the other agents' actions. That is , $\forall a_i \in A_I$:

$$u^i(a_1^*, \ldots, a_{i-1}^*, a_i^*, a_{i+1}^*, \ldots, a_n^*) \geq u^i(a_1^*, \ldots, a_{i-1}^*, a_i, a_{i+1}^*, \ldots, a_n^*)$$

Let consider the case with two agents.

Proving that a set of strategies is an equilibrium To check that (a_1^*, a_2^*) is an equilibrium, it is enough to verify that there is no unilateral deviation increasing the deviating agent 's payoff

Checking for all the equilibria of a game In a finite game, there is a

Analyzing different equilibria of some dynamic game

Consider the normal form representation of the dynamic game presented some slides before

1 \ 2	ll	lr	rl	rr
L	$1,2$	$1,2$	$3,4$	$3,4$
R	$5,6$	$1,2$	$5,6$	$1,2$

Player 2 's rationality, when he anticipates that player 1 plays L is to omit all L line 's cells in which player 2 's payoff is dominated (hatched in green).

L	$1,2$	$1,2$	$3,4$	$3,4$

Player 's 2 rationnality is complete when looking at all anticipations in $\{L, R\}$. (see green hatched areas.) Then, Player 1 's rationality, for each anticipation of what player 1 plays (in $\{ll, lr, rl, rr\}$) is to omit in each considered column, the cells in which player 1 's payoff is dominated (hatched in Blue).

1 \ 2	ll	lr	rl	rr
L	$1,2$	$1,2$	$3,4$	$3,4$
R	$5,6$	$1,2$	$5,6$	$1,2$

- Check directly that (R, ll), (R, rl), (L, rr) are three Nash equilibria.

Let check that (R, ll) is a NE. We draw only the relevant information from the preceding table

2 / 1	ll	lr	rl	rr
L	$1, 2$			
R	$\mathbf{5}, \mathbf{6}$	$1, 2$	$5, 6$	$1, 2$

- *Is there a profitable deviation for Player 1?* NO. Indeed, instead of playing R, she would play L, obtaining $1 \leq 5$.

- *Is there a profitable deviation for Player 2?* NO. Indeed, instead of playing ll, she could play lr, obtaining $2 \leq 6$, she could play rl, obtaining $6 \leq 6$ or she could play rr, obtaining $2 \leq 6$. Any case, she cannot increase her profit at the deviation stage.

- *We conclude that (R,ll) is a Nash Equilibrium*

Example of NE without dominant strategy

Consider the case where there are two firms which marginal cost is constant, equal to c, q_1 is firm 1's rate of production while q_2 is firm 2 's rate of production. The demand market is represented by

$$p = \alpha - (q_1 + q_2)$$

- Prove that the unique equilibrium of this production game is $q_1 = q_2 = (\alpha - c)/3$.
- Are the equilibrium strategies of this game dominant ?

Examples of NE

Compute Nash Equilibrium for each following game When the strategy space for the players are $S_1 = S_2 = [0, 1]$ and with the pay-off functions :

a) ▶ $$g_1(x, y) = 5xy - x^2 - y^2 + 2$$
$$g_2(x, y) = 5xy - 3x^2 - 3y^2 + 5$$

b) ▶ $$g_1(x, y) = 5xy - x - y + 2$$
$$g_2(x, y) = 5xy - 3x - 3y + 5$$

c) ▶ $$g_1(x, y) = -2x^2 + 7y^2 + 4xy$$
$$g_2(x, y) = (x + y - 1)^2$$

d) ▶ $$g_1(x, y) = -2x^2 + 7y^2 + 4xy$$
$$g_2(x, y) = (x - y)^2$$

Best responses

Nash equilibrium consideration sheds light on Bests responses.

Given appropriate differentiability assumptions, and assuming the a_i's are single-dimensional so that A_I is a subset of the real line, the Nash equilibrium $(a_1^*, \dots, a_n^*)$ if found. By solving n simultaneous equations

$$\frac{\partial u_i}{\partial a_i} = 0 \quad \text{for all } i = 1, \dots, n \tag{1}$$

Definition

In game theory, the best response is the strategy (or strategies) which produces the most favorable outcome for a player, taking other players' strategies as given

Rationalizability and iterated dominance

In this solution concept, players are assumed to be rational and so strictly dominated strategies are eliminated from the set of strategies that might feasibly be played. A strategy is strictly dominated when there is some other strategy available to the player that always has a higher payoff, regardless of the strategies that the other players choose.

- In game theory, rationalizability is a solution concept. The general idea is to provide the weakest constraints on players while still requiring that players are rational and this rationality is common knowledge among the players. It is more permissive than Nash equilibrium. Both require that players respond optimally to some belief about their opponents' actions, but Nash equilibrium requires that these beliefs be correct while rationalizability does not. Rationalizability was first defined, independently, by Bernheim (1984) and Pearce (1984).

- It can be easily proved that every Nash equilibrium is a rationalizable equilibrium ; however, the inverse is not true. Some rationalizable equilibria are not Nash equilibria. This makes the rationalizability concept a generalization of Nash equilibrium concept.

Example : the matching pennies

As an example, consider the game matching pennies pictured below. In this game the only Nash equilibrium is row playing h and t with equal probability and column playing H and T with equal probability. However, all the pure strategies in this game are rationalizable.

Each of the two players has a penny. They independently choose to display either heads or tails. If the two pennies are the same, player 1 takes both pennies. If they are different, player 2 takes both pennies.

	H	T
h	1,-1	-1,1
t	-1,1	1,-1

Consider the following reasoning : row can play h if it is reasonable for her to believe that column will play H. Column can play H if its reasonable for him to believe that row will play t. Row can play t if it is reasonable for her to believe that column will play T. Column can play T if it is reasonable for him to believe that row will play h (beginning the cycle again). This provides an infinite set of consistent beliefs that results in row playing h. A similar argument can be given for row playing t, and for column playing either H or T.

Iterative elimination of dominated strategies

- The iterative procedure of dominated strategies makes a bunch of assumptions :
  - Not only we suppose that each player is rational
  - but also, each player should anticipate that the other players will play rationally, and conform to this iterative procedure

3. Existence and efficiency

Existence and/of multiplicity : two basic examples

▶**Rock–paper–scissors** Ciseaux coupent feuille, qui emballe pierre, qui casse ciseaux

	P	C	F
p	0,0	-1,1	+1,-1
c	-1,+1	0,0	+1,-1
f	+1,-1	-1,+1	0,0

No equilibrium

▶**Battle of the sexes** A husband and wife want to go to movies. They can select between “Devils wear Parada” and ”Iron Man”. They prefer to go to the same movie, but while the wife prefers “Devils wear Parada” the husband prefers “Iron Man”. They need to make the decision independently.

	DP	IM
dp	3,2	1,1
im	1,1	2,3

Two equilibria

Existence of a NE

NE is linked to the existence of a solution of a fixed point of the correspondance of bests responses. Point fixe theorems allow to characterize games for which an equilibrium does exists. The more frequent result is the following :

Existence Theorems

1. In a finite game, a Nash equilibrium always exists, at least in mixed strategies ;
2. In the general case, there exists a Nash equilibrium
   - whenever strategy spaces are convex and compacts
   - Whenever the. Payoff functions are quasi-concave and continuous.

Mixed strategies Nash equilibrium

In certain environments, Mixed strategies seem to be more natural than pure strategies. This is the case, for exemple, when we analyze Penalties in football.

A mixed strategy assigns a probability distribution over pure strategies. Formally, a mixed strategy of agent i , $\sigma_i \in \Delta(Ai)$, defines a probability, $\sigma_i(ai)$ for each pure strategy $ai \in Ai$.

In this view, a pure strategy is a special case of mixed strategy, associated with a degenerate distribution.

Penalty game, entre le goal (1) et le buteur (2)

	$p(G) = \beta$	$p(D) = 1 - \beta$
$p(g) = \alpha$	1,-1	-1,1
$p(d) = 1 - \alpha$	-1,1	1,-1

- Find the parameters α and β such that the corresponding mixed strategies form an equilibrium.

Comparison of NE with a Pareto optimum

At a Pareto optimum, no agent can be made better off without some other agent being made worse off. That corresponds to the maximization of a weighted sum of the agent's utility $\sum_j \omega_j u^j$ wich first order conditions are :

$$\sum_{j=1}^{n} \omega_j \frac{\partial u_j}{\partial a_i} \qquad \text{for all} i = 1, \ldots, n \tag{2}$$

- We examine in the following a particular case whenever there are large conflicts between the agents' objectives.

Inefficiency of NE When there are too much conflicts

Suppose that at a Pareto optimum, the different agents' interests conflict, in the sens that if anything agents were to increase his action a_I, this. Would increase his own utility but lower every other agents's utility, that is :

$$\left.\frac{\partial u_i}{\partial a_i}\right|_{u^*} \geq 0 \text{ and } \left.\frac{\partial u_j}{\partial a_i}\right|_{u^*} > 0 \text{ for all } j \neq i \tag{3}$$

This equation means that we are considering that the Pareto frontier is negatively sloped

Exercise : make a picture representing (3)

- THEN, it follows from (3) and (2) that at a PO, $\left.\frac{\partial u_I}{\partial a_i}\right|_{u^*} > 0$, which means that <u>(2) and (1) cannot be satisfied simultaneously</u> : the Nash equilibrium is not Pareto optimal.

Student goal : the ability to transformer a story in a game

- A pedestrian who does not cross at the zebra crossing in a country where motorists engage in uncivilized behavior solves a basic problem in game theory, as it correctly anticipates the likely behavior of others. Conversely, the same is true for the unkind motorist who expects that a pedestrian will not cross when his car is approaching a zebra crossing, given the bad reputation of motorists in the country. Ultimately, each of us is daily involved in situations falling within the game theory.

- Describe a corresponding game (defining mixed strategy) that describes this situation between the pedestrian and the motorist, by calling p the probability that the pedestrian crosses the zebra crossing as the car approaches and q la probability that the car does not stop at the zebra crossing.

The payoffs (pedestrian, car) are as follows : if the pedestrian crosses and the car passes (-1, *alpha*) ; if the pedestrian crosses and the car does not pass (1.0) ; if the pedestrian does not cross and the car passes (0,1) ; if the pedestrian does not cross and the car does not pass (0, 0).