Resolution rules in a system of financially linked firms

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Abstract The reimbursement abilities of firms holding liabilities on each other are intertwined, potentially generating coordination failures and defaults through uncontrolled contagion. In stress episodes, these linkages thus call for an orderly resolution, as implemented by a regulatory authority assigning the amount each firm within the system reimburses to each other one. The paper studies such resolution by considering 'rules', assuming their primary goal is to avoid default on external debts, say, banks' defaults on deposits. The main objective is to investigate what proportionality means for a rule, taking into account various legal and informational constraints.

Keywords cross-liabilities, defaults, resolution, proportionality, entropy

JEL classification D71, G33

1 Introduction

Firms, especially in the financial sector, hold liabilities on each other. As a result, their reimbursement abilities are intertwined, thereby potentially generating coordination failures and defaults through uncontrolled contagion. In stress episodes, these linkages thus call for an orderly resolution, as implemented by a regulatory authority assigning the amount each firm within the system reimburses to each other one. Orderly resolutions arise in practice in discretionary or systematic ways. Given the risk posed by a bankruptcy of LTCM, the Federal Reserve Bank of New York organized a bailout by major creditors. More systematically, a central counterparty (CCP) organizes the liquidation of the liabilities of a defaulting member and allocates the losses among the other members. The benefit of fast and orderly liquidation of complex and intertwined positions became salient in the Lehman Brothers' bankruptcy[2] and

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[2]Lehman Brothers' positions cleared by CCPs were liquidated fast (Fleming and Sarkar 2014).

partly motivated new regulations.[3] The liquidation of cross-liabilities -instead of the liabilities of a single firm- raises new design issues related to contagion, the netting of positions, the proportionality and fairness of reimbursements made and received. The objective of this paper is to investigate such issues in a stylized model of financial linkages. The analysis relies on resolution rules and their properties. Rules contrast with disorderly or discretionary liquidations of conflicting claims and describe the perspective of a regulator (or an exchange operator) who clarifies how conflicting claims will be solved in case a default arises.

We consider a system of entities, called hereafter firms, which have claims and liabilities between themselves, all with equal priorities. Firms also have relationships with entities external to the system, resulting in a net external value, which can be positive or negative.[4] The analysis takes place at a single liquidation date. We assume that default by a firm on external entities, say default by a bank on customers' deposits, triggers bankruptcy, while default on firms within the system does not. Call *distressed* a firm whose net external value is less than its net internal liabilities. Requiring a distressed firm to fully reimburse its net internal liabilities makes it default on its external debt hence bankrupt, raising a first question: Is it possible to allocate the limited resources of the distressed firms so as to avoid their bankruptcy without causing that of their non-distressed creditors? Such allocation is described by a *solution*, which defines (partial) reimbursements within the system so that no firm is bankrupt, i.e. the net worth of each is non-negative. Solutions may not exist, meaning that bankruptcy cannot be avoided without bail-out. When they do exist, solutions may be numerous, raising a second question: How to select a solution? On what principle should the reimbursements be based? Proportionality is such a principle. If a single firm is indebted and its creditors are safe, proportionality simply means that the firm reimburses the same amount per unit of claim to each creditor. However, in a system composed of multiple firms that are simultaneously debtors and creditors, and when furthermore reimbursements face various constraints stemming from law or common practice, what proportionality means is unclear. Our aim is

[3]The Dodd-Frank Act (2010) for the USA and the European Market Infrastructure Regulation (EMIR 2012) for the EU require the use of CCPs for a large class of derivatives.

[4]The model is similar to Eisenberg and Noe (2001) except that here firms may be indebted to entities outside the system.

precisely to investigate proportionality of resolution rules in these contexts.

A (resolution) rule is defined for a range of liquidation problems. It specifies a solution for each problem that may realize before knowing which one will realize. The information available at the resolution date determines which solutions are admissible from a legal and regulatory perspective and from the firms' point of view. We always assume that the regulator knows the net external value of each firm as well as its total internal liabilities and claims within the system, thus, in particular, knows which firms are distressed. As for bilateral liabilities, two settings are investigated. In the first one, the regulator does not know the bilateral values[5] and selects a *coarse* solution. Such a solution specifies the total amount paid and received by each firm on the basis of external values and claims and liabilities' totals. These amounts can be understood as sent to and dispatched by the regulator. In the second setting, the regulator knows all the bilateral values and selects a *full* solution. Such a solution specifies the reimbursement of each firm to each other one on the basis of full information. We define a variety of 'constrained-proportional' (CP) solutions, depending on the information setting and the imposed constraints. One type of constraints bounds how much a firm receives or reimburses, either at the total or bilateral level. For example, requiring that a firm never receives more than its claims excludes bail-out within the system; also, under full information, capping bilateral reimbursements by the corresponding liabilities is a common practice. Another type of constraints aims at minimizing the impact of distressed firms. For example, some legislations require distressed firms to be fully repaid or non-distressed firms to reimburse fully their liabilities.[6] Adding constraints however limits the possibility of resolution. A first task is to determine for which external values and cross-liabilities a solution satisfies all the required constraints. For this we rely on flow analysis in networks.

Two approaches characterize CP-solutions. The first one is based on a measure that evaluates the inequality of the transfers relative to the due ones. We rely on the entropy measure: CP-solutions minimize the entropy measure over the solutions satisfying the required constraints. The second approach, called axiomatization, eval-

[5]Or the regulator knows them but voluntarily dismisses the information, as in the case of a CCP.

[6]For the Lehman Brothers' resolution, creditors who lost their claims on the US branch had nevertheless to reimburse their liabilities to the firm, but compensation was possible in the UK.

uates a rule through 'properties' (axioms) that the regulator considers as desirable. Properties may bear on the solution assigned to a problem or on the rule's behavior, i.e. on how the assigned solution changes when the data of the problem varies. Considering rules allows one to address questions such as: Can a firm be penalized for earning more from outside? Can the other firms be penalized? Negative answers to these questions impose some form of monotony in the assigned solutions. The three main properties used to characterize the CP-solutions are monotony, proportional target and creditors' priority.

The full CP-solutions are built on two indices per firm, which reflect proportionality in each direction, reimbursements made and payments received. Consider firm i. i's *rescue index* adjusts up each claim of i if this is necessary to avoid its bankruptcy. i's *reimbursement ability* index is determined by creditor's priority and specifies the common proportion by which i reimburses each of its bilateral liabilities adjusted by the creditor's rescue index; i's ability index thus depends on the composition of its debt through the need to reimburse more some creditors than others (which occurs if some have a rescue index larger than 1). In turn, the ability indices determine the payment received hence the need to be rescued: indices are interdependent. When bilateral reimbursements may exceed the corresponding liabilities, the CP-solution is bi-proportional, meaning that each reimbursement is equal to the liability multiplied by the borrower's ability and creditor's rescue index. When reimbursements are capped by liabilities, the CP-solution is a truncated version of a bi-proportional solution. Whatever case, the reimbursement of a liability is affected by both the borrower's reimbursement ability and creditor's rescue index, implying that the composition of a firm's debt affects its reimbursements. Coarse CP-solutions are also built on rescue indices, one for each firm, but the reimbursement index is common to all, reflecting the overall capacity of the regulator to reimburse adjusted claims. As a result, the composition of a firm's debt has no effect on its reimbursements.

Related literature. The paper is related to several strands of the literature. A first strand addresses the adjudication of conflicting claims when an estate must be divided among claimants whose claims' total is larger than the estate. This 'simple' claims problem introduced by O'Neill (1982) arises in a large number of situations, ranging from inheritance, tax allocation (Young 1987) and default by a single firm. Allocating the estate in proportion of the claims is a widely used rule (provided there are no specific constraints). Other rules, such as the 'Talmud' rule suggested by the Talmud, have been defined and axiomatized (Aumann and Maschler (1985) and Thomson (2003) survey). Our model follows a similar approach in a much more general setting where firms hold claims on each other and face bankruptcy constraints. In such a cross-liabilities setting, Eisenberg and Noe (2001) define a rule (hereafter EN rule) when the firms' external values are positive. The EN rule requires a firm to reimburse each of its creditors in proportion of their claims independently of their health (see Csoka and Herings (2018) for an axiomatization). When some external values are negative, such proportionality in reimbursements may trigger bankruptcy avoidable by using the CP-rule, which adjusts reimbursements to the creditors' health. In practice, proportionality in reimbursements is not satisfied (for example, in 2011, private creditors accepted a 50 percent loss on their Greek bonds). Based on the EN rule, Rogers and Veraart (2013) introduce costs to default on internal debts and show that the stockholders of a pool of firms may benefit from merging with a failing bank when these costs are high enough. Our setting differs since default within the system does not involve costs but those on external costs are (implicitly) very high and the transfers are decided by the regulator. Finally, Stutzer (2018) and Schaarsberg, Reijnierse and Borm (2018) study how to extend some of the rules defined in simple claims problems to those with cross-liabilities (assuming positive external values). Using a consistency axiom, the latter paper shows that extended solutions exist but are not unique, except for a specific structure of liabilities called 'hierarchical'. This is in line with our analysis where the same consistency axiom is too weak to pin down the constrained proportional solutions.

A second strand of the literature evaluates the fairness of an allocation of resources or losses (in the case of taxation) by computing a measure (often called index) meant to reflect a distance to a kind of ideal. Such approach applies quite generally, for example for measuring the segregation of students' assignment to schools (Frankel and Volij 2011). There are a variety of measures: the Gini index, the family of Atkinson's indices, the Mutual Information index, and the entropy one used here.

Balinski and Demange (1989-a) and recently Moulin (2016) use entropy to define proportionality in two-dimensional settings under various constraints (see the latter paper for further references).

Finally, a large empirical literature evaluates the effect of cross-liabilities on defaults and systemic risk by conducting simulations on calibrated systems, relying on the EN rule (see the survey of Upper (2011) who also discusses the issue of missing information on bilateral values) or on an exogenous process of propagation of defaults through balance-sheets (see e.g. Gai and Kapadia (2010)).

Section 2 introduces the model, defines solutions and their properties, and describes the simple claims problem and EN rule. Sections 3 and 4 study coarse and full rules. Section 5 gathers the proofs.

2 Solutions in a financially linked system

Consider a system composed of n firms, say banks and intermediaries in a financial system, or countries in an integrated market. Denote $N = \{1, \cdots, n\}$. At a liquidation date, firms have claims and liabilities between themselves and with entities outside N. Firm i's nominal liability to firm j is represented by non-negative ℓ_{ij} (equivalently ℓ_{ij} is j's claim on i) where by convention ℓ_{ii} is null. Denote $\boldsymbol{\ell} = (\ell_{ij})_{i,j=1,\cdots,n}$ and ℓ_{Ni} and ℓ_{iN} i's total claims and liabilities: $\ell_{Ni} = \sum_{j \in N} \ell_{ji}$ and $\ell_{iN} = \sum_{j \in N} \ell_{ij}$.[7] The liabilities within N all have the same priority. The liability graph $\mathcal{G}$ represents the positive liabilities within N: (i, j) is in $\mathcal{G}$ if $\ell_{ij} > 0$. Each firm i in N has also claims (stocks, loans..) on entities outside N, with value a_i, and liabilities to them (debts, deposits), with value d_i. i's *net external value* is defined by $z_i = a_i - d_i$.

An allocation describes the (possibly partial) reimbursements within N. We consider two types of allocations. A *full allocation* specifies the non-negative transfer b_{ij} from each firm i to each other one j. To fix the terminology, b_{ij} is referred to i's reimbursement to j or j's payment by i. Denote $\boldsymbol{b} = (b_{ij})_{i,j=1,\cdots,n}$ where by convention $b_{ii} = 0$. A *coarse allocation* specifies the total reimbursement and total received payment by each firm, denoted respectively by b_{iN} and b_{Ni} for firm i,[8] which sat-

[7] We use throughout the following notation. For a matrix $\boldsymbol{x} = (x_{ij})$ and two subsets A and B of the rows and columns' indices, we denote $x_{A,B} = \sum_{i \in A, j \in B} x_{ij}$ (the comma is dropped when there is no confusion). Similarly, for a vector $\boldsymbol{x} = (x_i)$, $x_A = \sum_{i \in A} x_i$.

[8] Although these totals are not computed as sums of bilateral transfers, it is convenient to use

isfy the balance condition: $\sum_i b_{iN} = \sum_i b_{Ni}$. Denote $\boldsymbol{B} = (b_{iN}, b_{Ni})_{i=1,\cdots,n}$. Natural allocations are the *exact* ones, which specify no default within N: the coarse exact allocation is defined by $b_{Ni} = \ell_{Ni}$ and $b_{iN} = \ell_{iN}$ for each i and the full exact one by $b_{ij} = \ell_{ij}$ for each i, j.

The information available to the regulator conditions which types of allocations make sense and which constraints may be imposed on the allocations, as described below. We always assume that the regulator knows the net external values, as well as the total value of the internal liabilities and claims of each firm. If the regulator's information is limited to these values, the problem is coarse described by $\Pi = (z_i, \ell_{iN}, \ell_{Ni})_{i=1,\cdots,n}$ where totals are balanced: $\sum_i \ell_{iN} = \sum_i \ell_{Ni}$. In that case, the regulator can reasonably only choose coarse allocations. If the regulator knows furthermore all bilateral liabilities, the problem is described by the full set of data: $\pi = (\boldsymbol{z}, \boldsymbol{\ell})$. In that case, the regulator can choose coarse or full allocations.

Default on external creditors triggers bankruptcy. Thus, i is *not bankrupt* at an allocation ($\boldsymbol{B}$ or $\boldsymbol{b}$, see footnote 8) if $a_i + b_{Ni} \geq b_{iN} + d_i$, or equivalently, if *$i$'s net worth W_i* is non-negative where $W_i = z_i + b_{Ni} - b_{iN}$. The condition stems from limited liability, according to which stockholders cannot be forced to inject cash. If each firm has a non-negative nominal net worth, $z_i + \ell_{Ni} - \ell_{iN} \geq 0$ for each i, no firm is bankrupt at the exact allocation. On the contrary, a firm that has a negative nominal net worth, called *distressed*, goes bankrupt at the exact allocation. Avoiding its bankruptcy inflicts a loss to the other firms in N.[9] As a result, some non-distressed firms may end up in difficulty and default on their internal liabilities or even go bankrupt through contagion effects. Solutions and rules, defined next, reflect that the primary objective of the regulator is to avoid bankruptcy, while preserving reimbursements within the system as much as possible, in particular by choosing the exact allocation provided no firm is bankrupt. Additional constraints reflecting law or common practice will be imposed later on, thereby affecting which problems admit a

the same notation as for the totals associated to a full allocation $\boldsymbol{b}$. In particular, the definitions that bear on reimbursements and payments only through their total for each firm and not on the bilateral values are valid for coarse or full solutions by having $\boldsymbol{B} = (b_{iN}, b_{Ni})_{i=1,\cdots,n}$ either a coarse allocation or the totals associated to a full allocation $\boldsymbol{b}$.

[9] $z_i + b_{Ni} - b_{iN} \geq 0$ implies that i's net reimbursement is lower than its net liabilities: $\ell_{iN} - \ell_{Ni} > b_{iN} - b_{Ni}$.

solution. This explains why the domain $\mathcal{F}$ on which a rule is defined is left unspecified in the following definition.

Definition 1 *Consider a problem* Π *or* π*. The set of distressed firms, denoted by* D*, is* $D = \{i, z_i + \ell_{Ni} - \ell_{iN} < 0\}$*. A solution is an allocation* $\boldsymbol{B}$ *or* $\boldsymbol{b}$ *for which no firm is bankrupt:*

$$W_i = z_i + b_{Ni} - b_{iN} \geq 0 \text{ for each } i. \tag{1}$$

A rule F *on a domain* $\mathcal{F}$ *assigns to each problem in* $\mathcal{F}$ *a solution, which must be the exact solution when there are no distressed firms, i.e. when the set* D *is empty.*

To illustrate the setting consider the following example, which will be used repeatedly.

Example 1 $z_1 = -1$, $z_2 \geq \frac{1}{4}$, $z_3 = \frac{3}{4}$ and the liabilities matrix

$$\boldsymbol{\ell} = \begin{pmatrix} 0 & 1 & a \\ a & 0 & 1 \\ 1 & a & 0 \end{pmatrix} \quad \text{where } 0 < a < 1.$$

For $z_2 \geq \frac{1}{4}$, a solution exists (see the next paragraph). Total liabilities and total claims are equal across the firms and 1 is distressed. Assume $z_2 = \frac{3}{4}$. Under coarse information, firms 2 and 3 cannot be distinguished, so the regulator should reasonably assign them identical transfers. Under full information, distressed 1 is known to have borrowed more from 2 than from 3, and 3 to have borrowed more from distressed 1 than from 2. A regulator may use this information and assign different totals, thereby affecting differently 2 and 3's net worths. ■

It is easy to see that a problem admits a solution if the system is not globally indebted, i.e. if $\sum_i z_i \geq 0$. The condition is necessary due to the *conservation of aggregate net worth*: because the transfers within N cancel out, $\sum_i W_i$ is equal to the aggregate external value $\sum_i z_i$ at any allocation. Hence $W_i \geq 0$ for each i requires $\sum_i z_i \geq 0$. The condition is sufficient[10] and, furthermore, there are many solutions if $\sum_i z_i > 0$. In that case, the sole no-bankruptcy conditions (1), which depend on the external values only, leave a lot of flexibility in choosing the transfers. We now introduce constraints to relate transfers to the claims and liabilities. These constraints

[10] For example, for the firms with $z_i < 0$, set $b_{Ni} = -z_i$ and $b_{iN} = 0$. For the firms with $z_i \geq 0$, set $b_{Ni} = 0$ and b_{iN} such that $z_i \geq b_{iN}$ and $\sum_{i,z_i \geq 0} b_{iN} = -\sum_{i,z_i<0} z_i$. The latter conditions are compatible since $\sum_i z_i \geq 0$ writes $-\sum_{i,z_i<0} z_i \leq \sum_{i,z_i \geq 0} z_i$. These transfers are balanced and no firm is bankrupt: solutions exist and there are numerous $\sum_i z_i > 0$.

are not required throughout. One goal of the analysis is precisely to determine their implications, in particular the problems for which they are compatible and the shape of the solutions.

Constraints on totals We define constraints that bear on the totals reimbursed and paid by each firm in relation with their total claims and liabilities. They apply to a coarse solution or to the totals of a full one.

Bounds Reimbursements are bounded if no firm reimburses more in total than its liabilities' total: $b_{iN} \leq \ell_{iN}$ for each i. Payments are bounded if no firm receives more in total than its claims' total: $b_{Ni} \leq \ell_{Ni}$ for each i. For short, a solution is said to be *R&P-bounded* if both reimbursements and payments are bounded.

Bounded payments exclude a bail-out within the system. At a solution, they result in distressed firms defaulting on their liabilities to N: $b_{iN} \leq z_i + \ell_{Ni}$ implies $b_{iN} < \ell_{iN}$ for $i \in D$.

Creditors' priority (over stockholders) requires that a firm with positive net worth fully reimburses its liabilities, i.e. for each i: either $b_{iN} = \ell_{iN}$ or $z_i + b_{Ni} - b_{iN} = 0$.

In practice, *R&P*–boundedness and creditors' priority are often imposed. They prevent coordination failure without further regulators' intervention if each firm has a positive nominal worth,[11] as stated in the following property:

Property 1 *Let the net nominal worth of each firm be positive. The exact coarse solution is the unique one that satisfies R&P–boundedness and creditors' priority.*

We will mainly consider solutions that satisfy additional constraints meant to minimize (in some sense) the distortions posed by distressed firms:

Definition 2 *A solution* $\boldsymbol{B}$ *to a problem is said to be* **tight** *if it is R&P-bounded and satisfies*

$$\text{for each } i \in D: \quad b_{Ni} = \ell_{Ni} \quad \text{(full payments to distressed)} \tag{2}$$

$$\text{for each } i \in D^c: \quad b_{iN} = \ell_{iN} \quad \text{(full reimbursements by non-distressed)} \tag{3}$$

[11] This assumption is slightly stronger than assuming no distressed firms. If no firm is distressed but some have null nominal worths, Property 1 does not hold. For example, consider two firms owing the same amount to each other, say 1 unit, and with null external values. Their nominal net worths are null. Many solutions satisfy *R&P*–boundedness and creditors' priority: let each reimburse the same amount b to the other with b less than 1.

It is said **super-tight** *if in addition the distressed firms' net worth is null:*

$$\text{for each } i \in D : z_i + \ell_{Ni} = b_{iN} \qquad \text{(minimal rescue)} \tag{4}$$

The tightness conditions can be interpreted as tools for limiting contagion within the system: (2) require distressed firms to be fully repaid (even if they default) and (3) require non-distressed firms to fully reimburse their liabilities total (even if they are not fully repaid). In addition, minimal rescue constrains the loss due to the rescue of distressed firms to be minimal (since rescue implies $z_i + \ell_{Ni} \geq b_{iN}$). A super-tight solution satisfies creditors' priority since the non-distressed firms do not default.

Bilateral constraints Under full information, the bilateral liabilities are known, calling for assigning bilateral transfers $\boldsymbol{b}$ related to them. A natural requirement is that a firm makes a positive transfer only to a creditor. A stronger one is that this transfer is capped by the corresponding liability. Formally:

Full allocation $\boldsymbol{b}$ is *liability-compatible* if for each $i, j : b_{ij} > 0$ only if $\ell_{ij} > 0$, equivalently only if $(i, j) \in \mathcal{G}$. Allocation $\boldsymbol{b}$ is *bilaterally-bounded* if for each $i, j : \; b_{ij} \leq \ell_{ij}$. Clearly, a bilaterally-bounded allocation is liability-compatible. It is easy to check that Property 1 extends as follows: The exact full solution is the unique solution that is bilaterally-bounded and satisfies creditors' priority.

We now provide two illustrations of the setting.

Illustration 1: Simple claims problem (O'Neill 1982) In a simple claims problem, an amount T has to be divided among a group J of claimants, each one having a claim, c_j for j, where $\sum_j c_j \geq T$. In addition, each claimant j must receive a minimum m_j, $m_j \geq 0$, which defines j's bankruptcy constraint. The problem is thus described by $\sigma = (J, T, \mathbf{c}, \mathbf{m})$. In our model, a simple claims problem obtains if a single firm is indebted within the system.[12] Furthermore, as shown in Section 3, general problems are reduced to simple claims problem under some conditions.

The next definition introduces the constrained proportional rule, which extends the proportional rule to account for the minimal requirement.[13]

[12] If 1 is the indebted firm, $J = N - \{1\}$ and $c_j = \ell_{1j}$ for j in J. Assuming creditor's priority and bounded reimbursements, necessarily $T = \min(z_1, \ell_{1N})$. Set $m_j = \max(-z_j, 0)$ for $j \in J$ to avoid j's bankruptcy.

[13] The literature on simple claims problems does not consider minimal requirements (i.e. implicitly $m_j = 0$) except Balinski and Young (1982). Their problem is to allocate a total number of seats T in a parliament to districts, given the population numbers $(c_1, \cdots, c_p)$ in the districts and a minimum

Definition 3 *Let* $\sigma = (J, T, \mathbf{c}, \mathbf{m})$ *be a simple claims problem. A solution* $\boldsymbol{x} = (x_j)_{j \in J}$ *to* σ *assigns an amount* x_j *to each claimant* j *such that* $x_J = T$ *and* $m_j \leq x_j$ *for each* j*.* σ *admits a solution iff* $m_J \leq T$*. In that case, the* **constrained-proportional** *solution, for short* **cp-solution**, *is the unique solution* $\boldsymbol{x}$ *for which there are positive* δ *and* $(\mu_j)_{j \in J}$ *such that for each* j *in* J*:*

$$x_j = \delta c_j \mu_j \text{ where } \mu_j \geq 1 \text{ with } \mu_j = 1 \text{ if } x_j > m_j.$$

Surely $\delta \leq 1$*. If* $m_j \leq c_j$ *for each* j *then* $\boldsymbol{x}$ *is payment-bounded:* $x_j \leq c_j$ *for each* j*. The cp-rule assigns to each problem its cp-solution.*

The cp-solution can also be written in the form $x_j = \max(\delta c_j, m_j)$ where δ is the unique value in]0,1] such that $\sum_j x_j = T$. Introducing the μ_j is useful for the sequel. The cp-rule can be described as follows. Assume to simplify the minimum required by each firm to be lower than its claim: $m_j \leq c_j$ for each j. Then, for $T = c_J$, each claimant receives its claim: $x_j = c_j$, setting δ and μ_j all equal to 1: The cp-solution is exact. Decrease T; the cp-solution assigns the proportional solution as long as no one is bankrupt: $\boldsymbol{x} = \delta \mathbf{c}$ for $\delta = \frac{T}{c_J}$ is the cp-solution as long as $\delta c_j \geq m_j$ for each j, setting each μ_j equal to 1. This reflects that, ideally, each firm should receive the same amount per unit of claim. Such a property is extended to general problems in Section 3, where it is referred to as Proportional payments' target. If some m_j are positive, decreasing T further, the proportional allocation hits the minimum for some j. In that case each claimant receives the same payment per unit of claim except those who would be bankrupt and receive relatively more: j receives δc_j except if $\delta c_j < m_j$, in which case the payment to j is scaled up by μ_j larger than 1. μ_j is qualified as j's *rescue index.*

The above description provides a characterization of the cp-rule (see Lemma 1 in Section 5): the cp-rule is the only rule that satisfies the ideal of proportionality described by Proportional payments' target together with a very natural monotony property with respect to the amount T, according to which no one's assigned amount goes down when the estate T goes up. Although the cp-rule almost does not need a justification, other rules reflect some idea of proportionality. One assigns to each j its

m_j for each j. Since the seats are not divisible, the allocation must be integer-valued. Considering the cp-solution to be the ideal one, Balinski and Young study how to transform it into integers.

minimum m_j and allocates the remaining, $T - \sum_{i \in J} m_j$ (which is non-negative for a feasible problem) in proportion of the residual claims if these are positive.[14] This rule assigns proportionally a smaller amount than the cp-one hence harms (resp. benefits) those who have a large (resp. small) residual claim relative to others. It does not satisfy Proportional payments' target.

Illustration 2: Clearing reimbursement ratios (Eisenberg and Noe 2001). Assume that each firm reimburses the same fraction of its claims to each of its creditors: $b_{ij} = \tau_i \ell_{ij}$ where τ_i is i's reimbursement ratio, $0 \leq \tau_i \leq 1$. The vector $(\tau_i)_{i=1,\cdots,n}$ is said to be clearing if no firm is bankrupt and creditors' priority is satisfied:

$$\text{for each } i: \ \tau_i \ell_{iN} - \sum_j \tau_j \ell_{ji} \leq z_i \text{ with } \tau_i = 1 \text{ if the inequality is strict.}$$

When all external values are positive, a clearing vector exists and is unique, as shown by EN, thereby defining a rule that assigns bilaterally-bounded (but maybe not tight) solutions. When some external values are negative, there are problems for which any ratio vector triggers bankruptcy whereas bilaterally-bounded solutions exist. The following example illustrates this point.

Example 2 Consider a simple claims problem where 1 be the only indebted firm. At a clearing vector, 1 recovers nothing and reimburses $\tau_1 = min(\frac{z_1}{\ell_{1N}}, 1)$ per unit of liability to each creditor. Consider three firms, $z_1 = 1$, $z_2 = -0.7$, $z_3 = 2$ and $\ell_{12} = \ell_{13} = 1$. Hence $\tau_1 = 0.5$ and 2 and 3 receive 0.5 each. 2's net worth is -0.2: 2, who is not distressed, is bankrupt by 'contagion'. Bankruptcy is avoided at the cp-solution: setting $m_2 = 0.7$ and $m_3 = 0$ (see footnote 12), payments are $b_{12} = 0.7$ and $b_{13} = 0.3$, associated to $\delta = 0.3$, $\mu_2 = 7/3$ and $\mu_3 = 1$. ∎

3 Coarse rules

This section extends the cp-rule to the coarse setting with cross-liabilities. We first determine problems that admit coarse solutions satisfying some of the constraints introduced in Section 2.

[14] Assign $y_j = m_i + \gamma \max(0, c_j - m_j)$ to each j where γ is the unique value in (0,1) such that $\sum_{i \in J} \gamma \max(0, c_j - m_j) = T - \sum_{i \in J} m_j$. It is easy to show that $x_j > y_j$ if and only if $x_j = \delta c_j > m_j$, i.e. j is unconstrained at the cp-solution.

3.1 The existence and structure of super-tight solutions

The next proposition characterizes the problems that admit $R\&P$-bounded coarse solutions. Furthermore, it shows that these problems admit super-tight solutions, which are easily found by solving a simple claims problem. Let the *shortfall of the distressed firms* be defined by $S_D = -(z_D + \ell_{N,D} - \ell_{D,N})$; it is equal to the opposite of their (negative) nominal worths.

Proposition 1 *Coarse problem* Π *admits a* $R\&P$*-bounded solution if and only if*

$$\sum_i min(z_i, \ell_{iN}) \geq 0 \tag{5}$$

$$\text{and for each } i : z_i + \ell_{Ni} \geq 0. \tag{6}$$

Denote by $\mathcal{C}$ *the set of problems that satisfy (5) and (6). A problem* Π *in* $\mathcal{C}$ *admits solutions that are super-tight. At these solutions, payments to the non-distressed firms* $(b_{Nj})_{j\in D^c}$ *solve the simple claims problem* $\sigma = (D^c, T, \mathbf{c}, \mathbf{m})$ *where* T *is equal to their claims' total diminished of the shortfall of the distressed firms:* $T = \ell_{N,D^c} - S_D$*, and for each* j *in* D^c*:* c_j *equals* j*'s total claim* ℓ_{Nj} *and lower-bound* m_j *equals* $\max(\ell_{jN} - z_j, 0)$.

It is easy to see why conditions (5) and (6) are necessary for a $R\&P$-bounded solution to exist. Consider (5). i's no-bankruptcy requires $z_i \geq b_{iN} - b_{Ni}$, hence $\min(z_i, b_{iN}) \geq b_{iN} - b_{Ni}$. Summing these inequalities over all i, the balancedness condition implies $\sum_i \min(z_i, b_{iN}) \geq 0$. Consider (6). $z_i + \ell_{Ni}$ is an upper-bound to i's net worth, achieved when i reimburses nothing and receives its claims; hence it must be non-negative for i not to be bankrupt.

We show that (5) and (6) are sufficient by constructing a super-tight solution. At such a solution, only the payments to the non-distressed firms are flexible. To achieve super-tightness, the regulator must collect ℓ_{iN} from each non-distressed firm i, repay ℓ_{Ni} to each distressed i, who gives back $z_i + \ell_{Ni}$ (which is non-negative under (6) but less than ℓ_{iN}). This leaves the total $T = \ell_{D^c,N} + z_D$ for the payments to the non-distressed firms. T is exactly equal to the gap between their due claims ℓ_{N,D^c} and the shortfall S_D.[15] T must be allocated to the non-distressed firms so that no one is bankrupt. This requires $z_j + b_{Nj} \geq \ell_{jN}$ for each j in D^c (since j fully reimburses

[15] The identity $\ell_{D^c,N} + \ell_{D,N} = \ell_{N,D^c} + \ell_{N,D}$ implies $\ell_{D^c,N} + z_D = \ell_{N,D^c} + \ell_{N,D} - \ell_{D,N} + z_D$, which is equal to $\ell_{N,D^c} - S_D$.

its liabilities), hence the lower bound $m_j = \max(\ell_{jN} - z_j, 0)$ on b_{Nj}. This proves that the payments $(b_{Nj})_{j\in D^c}$ solve the simple claims problem σ. σ admits a solution which is furthermore payment-bounded because the feasibility condition $\sum_{i\in D^c} m_i \le T$ is equivalent to[16] (5) and m_j is less than j's claim ℓ_{Nj} since j is non-distressed. ■

The problem σ can be seen as allocating the shortfall to the non-distressed firms. σ admits many solutions if $\sum_{i\in D^c} m_i < T$. As a result, a rule selecting super-tight solutions may assign distinct solutions to two distinct problems that induce the same σ. For example, consider problems with four firms and identical data except for z_1, z_2, which may differ but sum to the same total and make both 1 and 2 distressed. These problems all induce the same simple claims σ problem on $\{3, 4\}$. An admissible rule assigns $b_{N3} = m_3$, $b_{N4} = T - m_3$ if $z_1 > z_2$ and $b_{N3} = T - m_4$, $b_{N4} = m_4$ if $z_1 \le z_2$. This does not make much sense, in particular because it is not known how much each 3 and 4 have lent to or borrowed from 1 and 2. The rule introduced in the next section does not have this drawback.

3.2 The coarse constrained proportional rule

According to Proposition 1, super-tight coarse solutions are characterized by solutions to simple claims problems, which determine the payments to the non-distressed firms. The CP- solutions are based on the cp-solution to these simple problems.

Definition 4 *Let* Π *be in* $\mathcal{C}$. *The* **(coarse) constrained-proportional** *solution, for short the* **CP-solution**, *is the unique super-tight solution* $\boldsymbol{B}$ *such that the payments* $(b_{Nj})_{j\in D^c}$ *to the non-distressed firms satisfy* $b_{N,D^c} = \ell_{N,D^c} - S_D$ *and there are positive* δ *and* $(\mu_j)_{j\in D^c}$ *for which*

$$\text{for each } j \in D^c : b_{Nj} = \delta \ell_{Nj} \mu_j \text{ and } \mu_j \ge 1 \text{ with } \mu_j = 1 \text{ if } W_j > 0. \tag{7}$$

The CP-rule assigns to each problem Π *in* $\mathcal{C}$ *its CP-solution.*

The CP-solution can be described as follows. Distressed firms are fully repaid their claims and reimburse the maximum to avoid bankruptcy. Non-distressed firms fully repay their liabilities. The amount $T = \ell_{N,D^c} - S_D$ collected by the regulator is

[16] It suffices to prove $T - \sum_{i\in D^c} m_i = \sum_i min(z_i, \ell_{iN})$. The identity $\max(\ell_{iN} - z_i, 0) + \min(\ell_{iN} - z_i, 0) = \ell_{iN} - z_i$ implies $m_i + \min(\ell_{iN}, z_i) = \ell_{iN}$, hence, summing over D^c: $\sum_{i\in D^c} m_i = \ell_{D^c,N} - \sum_{i\in D^c} min(z_i, \ell_{iN})$. This yields $T - \sum_{i\in D^c} m_i = z_D + \sum_{i\in D^c} min(z_i, \ell_{iN})$. This proves the claim since for i in D surely $\ell_{iN} - z_i > 0$ hence $z_i = min(z_i, \ell_{iN})$.

allocated to the non-distressed firms according to the cp-solution in the simple claims problem that account for their claims and bankruptcy constraints (compare conditions (7) with Definition 3). When the μ_j are all equal to 1, the payments are proportional to claims with a factor δ. When some μ_j are larger than 1, δ is the minimum payment per unit of claim and a firm j is repaid more ($\mu_j > 1$) only to avoid its bankruptcy. We call μ_j j's *rescue* index.

Example 1-b Consider Example 1. The regulator knows $z_1 = -1$, z_2, $z_3 = \frac{3}{4}$ and $\ell_{iN} = \ell_{Ni} = 1 + a$ for each i. Conditions (5) write $z_2 \geq \frac{1}{4}$. We have $S_D = 1$ and $T = 1 + 2a$. The CP-solution allocates T to 2 and 3 proportionally if both 2 and 3's worths are non-negative: it allocates $\frac{1}{2} + a$ to each if $W_2 = z_2 - \frac{1}{2} \geq 0$ (W_3 is positive equal to $\frac{1}{4}$). For $z_2 \leq \frac{1}{2}$, reimbursement's proportionality would make 2 bankrupt, so that 2 is repaid the minimal amount to avoid its bankruptcy: $b_{N2} = 1 + a - z_2$ and $W_2 = 0$. 3 is repaid what is left from T: $b_{N3} = a + z_2$, as long as 3's net worth $W_3 = \frac{3}{4} + a + z_2 - (1 + a)$ is non-negative i.e. $z_2 \geq \frac{1}{4}$, which is (5). ■

We characterize the CP-rule through two approaches, first by minimizing the entropy inequality measure, second by its properties (axiomatization).

Minimizing entropy The entropy measure on allocation $\boldsymbol{B}$ is defined by:

$$f(\boldsymbol{B}) = a \sum_i b_{iN} \left[log \left(\frac{b_{iN}}{\ell_{iN}} \right) - 1 \right] + (1 - a) \sum_i b_{Ni} \left[log \left(\frac{b_{Ni}}{\ell_{Ni}} \right) - 1 \right]$$

where a is a parameter between 0 and 1.

f measures a distance to proportional allocations. To illustrate, consider the set of allocations defined by a maximal amount S transferable across firms where S is lower than the total liabilities ℓ_{NN}: $\boldsymbol{B}$ satisfies $\sum_i b_{Ni} = \sum_i b_{iN}$ and $\sum_i b_{Ni} \leq S$. In that case f is minimized by setting reimbursements proportional to liabilities and payments proportional to claims: $b_{iN} = \delta \ell_{Ni}$ and $b_{Ni} = \delta \ell_{Ni}$ for $\delta = \frac{S}{\ell_{NN}}$.

In our model, allocations are subject to different constraints than an overall amount of transfers. To account for non-bankruptcy and $R\&P$-boundedness, we consider the following program:

$$\mathcal{P}: \text{ minimize } f(\boldsymbol{B}) \text{ over the } R\&P\text{-bounded solutions } \boldsymbol{B} \text{ of } \Pi.$$

Denote by $\mathcal{C}^*$ the set of problems that satisfy each inequality in (5) and (6) strictly.

Proposition 2 *Let* Π *be in* $\mathcal{C}^*$. *Whatever* a *in* $]0,1[$, *the CP-solution is the unique solution that solves* $\mathcal{P}$.

Proposition 2 states that the CP-solution minimizes entropy f over all $R\&P$–bounded solutions. Since the CP-solution is super-tight, minimizing entropy *implies* full reimbursements by non-distressed firms, full repayment to distressed ones and minimal rescue. Thus, under coarse information, super-tightness does not restrict the set of problems admitting a $R\&P$–bounded solution (Proposition 1) and furthermore is necessary for a $R\&P$–bounded solution to be as proportional as possible. The proof of Proposition 2 relies on the first order conditions of the Lagrangean of convex program $\mathcal{P}$ (which are necessary and sufficient for a problem in $\mathcal{C}^*$ because the feasible set of $\mathcal{P}$ has a non-empty relative interior). Up to a transformation, δ is associated to the multiplier of the balanced condition on transfers and the rescue indices μ_j to the non-negativity of j's net worth.

The axiomatic approach Let F denote a rule that assigns to each Π in $\mathcal{C}$ a coarse solution. We characterize the CP-rule by two properties. The first property, Proportional payments' target, bears on each solution assigned by F, $\boldsymbol{B} = F(\Pi)$. It states that firms should be repaid in proportion of their claims whenever this is possible. Specifically, consider a subgroup I of firms and assume that the total payment they receive at $\boldsymbol{B}$ is not proportional to their claims. Contemplate reallocating this total payment within I without changing their reimbursements. If such a reallocation does not make any one bankrupt, then it should be implemented. Note that the reallocation has no affect on firms outside the subgroup I. Formally:

Proportional payments' target *Solution* $\boldsymbol{B}$ *to* Π *satisfies proportional payments' target if the following holds: For each* I *subset of* N, *let* δ *be the ratio of the total payment received by* I *over the total of their claims:* $\delta = \frac{b_{NI}}{\ell_{NI}}$. *If* $z_i + \delta\ell_{Ni} \geq b_{iN}$ *for each* i *in* I, *then* $b_{Ni} = \delta\ell_{Ni}$ *for each* i *in* I.

The second property expresses the idea that no firm should be penalized by increases in net external values. Denote by $W_i(\Pi)$ i's net worth at the solution assigned by rule F to problem Π: $W_i(\Pi) = z_i + b_{Ni} - b_{iN}$ where $(b_{iN}, b_{Ni})_{i=1,\cdots,n} = F(\Pi)$.

Worths' monotony (with respect to $\boldsymbol{z}$) *Rule* F *is monotone if no one's net worth goes down when the external value of a firm goes up: Let* Π *and* Π' *be two identical* *problems except for j's external value. If $z_j < z'_j$, then $W_i(\Pi) \leq W_i(\Pi')$ for each i.*

Iteration of the property implies that if the external value of several firms increase, no one's net worth decreases.

Proposition 3 *The CP-rule is the unique rule on $\mathcal{C}$ that assigns tight solutions verifying proportional payments' target and that is worths' monotone with respect to $\boldsymbol{z}$.*

A rule satisfying the properties thus necessarily chooses a solution that satisfies minimal rescue. The proof goes as follows. Starting with external values for which no firm is distressed and the exact solution is assigned, let decrease those of the distressed firms up to the point where they become all distressed and their net worth becomes null at the exact solution. Decreasing further their external values, worths' monotony of F implies that their net worth must be null (hence minimal rescue) and that the total payment T to the non-distressed firms decreases. T is distributed according to a 'reduced' rule in a simple claims problem (Proposition 1). This reduced rule is monotone with respect to T (due to the monotony of F) and satisfies proportional payments' target, which characterizes the cp-rule (Lemma 1).

3.3 Netting

Netting positions is an increasingly used technique to decrease exposures. It can be implemented between two entities by cancelling positions in opposite directions, within a cycle (called compressing), or at a centralized level as performed by a central-counterparty (CCP). This section describes the impact of netting by a CCP on the feasibility of resolution.[17] Let N be the members of a CCP and $\boldsymbol{\ell} = (\ell_{ij})_{i,j=1,\cdots,n}$ their positions cleared by the CCP. Netting transforms $\boldsymbol{\ell}$ into coarse liabilities $\boldsymbol{L} = (L_{Ni}, L_{iN})_{i=1,\cdots,n}$ as follows

$$\begin{aligned} &\text{if } \ell_{Ni} - \ell_{iN} \geq 0 \quad (i \text{ is long}) \ : \quad L_{Ni} = \ell_{Ni} - \ell_{iN} \text{ and } L_{iN} = 0 \\ &\text{if } \ell_{Ni} - \ell_{iN} < 0 \quad (i \text{ is short}) \ : \quad L_{Ni} = 0 \text{ and } L_{iN} = \ell_{iN} - \ell_{Ni}. \end{aligned}$$

Let short firms reimburse less than their netted liabilities L_{iN}, long firms receive less than their netted claims L_{Ni}, and set all other transfers to zero. Such an allocation is $R\&P$-bounded for $\boldsymbol{L}$. It is possible to make no firm bankrupt if (and only if) the

[17]The impact of a CCP on resolution is far from being limited to netting in particular because a CCP asks ex-ante for contributions to a default fund that can be used in case of stress.

following conditions are satisfied, where N_- and N_+ denote the sets of short and long firms:

$$\textstyle\sum_{i\in N_-} \min(z_i, L_{iN}) + \sum_{i\in N_+} \min(z_i, 0) \geq 0 \tag{8}$$

$$\text{for each } i \in N_- : \; z_i \geq 0 \text{ and for each } i \in N_+ : \; z_i + L_{Ni} \geq 0. \tag{9}$$

These conditions follow from Proposition 1 applied to $\boldsymbol{L}$. They are stronger than those for the original liabilities. Condition (8) states that the maximum that can be reimbursed by short members, accounting for their resources and liabilities, $\sum_{i\in N_-} \min(z_i, L_{iN})$, covers the losses of the long members due to their external activities, $-\sum_{i\in N_+} \min(z_i, 0)$. (8) implies $\sum_i \min(z_i, \ell_{iN}) \geq 0$, the resource condition (5) without netting. Conditions (9) require no short firm to have a negative external value and no long one to be distressed for the original liabilities. They imply the corresponding conditions (6) without netting. Netting hampers some problems to admit solution because it reduces the flexibility in transfers: it has the same effect as forcing a short member to be fully repaid on its claims and a long member to fully reimburse its liabilities, *independently* of their distressed status.

Under minimal rescue, each short member i reimburses the amount $\min(z_i, L_{iN})$ to the CCP, which in turn redistributes the total $\sum_{i\in N_-} \min(z_i, L_{iN})$ to the long members so as to avoid their bankruptcy. Hence netting can be viewed as transforming the initial problem into a simple claims one where the CCP is the unique debtor and the claimants are the long firms. Although not made public, proportional repayments and firms' health are two main objectives for a CCP so that the CP-procedure is likely to be close to CCP's practice, meaning that long members are repaid $b_{Ni} = \max(-z_i, \delta L_{Ni})$ where δ is the unique value in]0,1] such that $\sum_{i\in N_+} \max(-z_i, \delta L_{N,i}) = \sum_{i\in N_-} \min(z_i, L_{iN})$.

4 Full resolution rules

This section extends the coarse constrained proportional rule to the full information setting when the regulator, knowing bilateral liabilities, uses a full rule that specifies bilateral transfers related to these liabilities. As defined in Section 2, a minimal relationship is liability-compatibility, according to which a firm makes a positive transfer to another one only if it is liable to it, and a stronger one is bilateral-boundedness, according to which the transfer is not greater than the liability. We start by examining which problems admit solutions satisfying one of these conditions (Proposition 4).

4.1 Existence of liability-compatible or bilaterally-bounded solutions

As a preliminary result, Property 2 states that tight and bilaterally-bounded solutions require each non-distressed firm to reimburse fully each of its liability and each distressed firm to have each its claims to be fully repaid.

Property 2 *Define $\mathcal{G}_D = \{(i,j) \in \mathcal{G}, i \in D, j \notin D\}$. At a tight and bilaterally-bounded solution $\boldsymbol{b}$, $b_{ij} = \ell_{ij}$ for any (i,j) not in $\mathcal{G}_D$; hence $\boldsymbol{b}$ writes $\boldsymbol{b} = (\boldsymbol{b}_{|\mathcal{G}_D}, \boldsymbol{\ell}_{|N^2-\mathcal{G}_D})$.*

The proof is straightforward: tightness requires full reimbursement by non-distressed firm i: $b_{iN} = \ell_{iN}$; adding bilateral bounds $b_{ij} \leq \ell_{ij}$ for each j implies $b_{ij} = \ell_{ij}$. Similarly, tightness requires full repayment to distressed firm j, $b_{Nj} = \ell_{Nj}$, which together with bilateral bounds imply $b_{ij} = \ell_{ij}$ for each j.

The next proposition uses the following definition. Given a subset A of N, let $D(A)$ be the set of debtors of A: $D(A) = \{i \in N$ such that there is $j \in A$ with $\ell_{ij} > 0\}$.

Proposition 4 *The necessary and sufficient conditions for π to admit*
(a) a R&P-bounded and liability-compatible solution are:

$$\text{for each } A, B \text{ subsets of } N: z_A + \ell_{A^c \cap D(B),N} + \ell_{N,A\cap B^c} \geq 0 \tag{10}$$

(b) a tight and liability-compatible solution:

$$\text{for each } A, B \text{ subsets of } N: z_A + \ell_{A^c \cap D(B),N} + \ell_{N,A\cap B^c} \geq \ell_{A\cap D^c \cap D(B)^c,N} + \ell_{N,D\cap A^c \cap B} \tag{11}$$

(c) a bilaterally-bounded solution:

$$\text{for each } A \text{ subset of } N: z_A + \ell_{A^c,A} \geq 0 \tag{12}$$

(d) a tight and bilaterally-bounded solution

$$\text{for each } A \text{ subset of } N: z_A + \ell_{A^c,A} \geq \ell_{A\cap D^c,A^c} + \ell_{A\cap D,A^c\cap D}. \tag{13}$$

The proposition is proved by defining a graph with lower and upper capacities for which the existence of a circulation is equivalent to the existence of a solution satisfying the required constraints. It is easy to see why conditions (12) are necessary for the existence of bilaterally-bounded solutions and why they differ from (13) when tightness is required. At a bilaterally-bounded solution, the left-hand side of (12) is a maximal amount accruing to A from entities outside A, composed of z_A (possibly negative) from entities outside N and $\ell_{A^c,A}$, when each entity outside A reimburses its liability to each one inside A. This maximal amount must be non-negative to avoid bankruptcy. When tightness is required, there are forced reimbursements from A to A^c, represented by the right hand side of (13): $\ell_{A\cap D^c,A^c}$ are those from the non-distressed firms to the distressed ones and $\ell_{A\cap D,A^c\cap D}$ from the distressed firms in A to the distressed ones (Property 2). Conditions (10) and (11) for the existence of liability-compatible solutions are interpreted similarly but more complex because maximal transfers from a subset to another one depend on other transfers. This explains why one needs to consider any A and B, with B possibly not subset of A,

Let us illustrate the impact of tightness in Example 1. The impact of bilaterally-boundedness instead of liability-compatibility is illustrated in Example 1-d.

Example 1-c Let $\sum_i z_i = z_2 - \frac{1}{4} \geq 0$. At any payment-bounded solution $z_1 + \ell_{N1} \geq b_{1N}$, which writes $a \geq b_{1N}$. Hence at a bilaterally-bounded and tight solution, 2 receives at most a from 1, a from 3 (Property 2) and reimburses $1 + a$, so that $W_2 \leq z_2 + 2a - (1 + a) = z_2 + a - 1$. The solution exists only if 2 is not bankrupt if $z_2 + a - 1 < 0$. If the solution is not required to be tight, let 2 reimburse only $z_2 + a$ to 3 instead of 1 so that $W_2 = 0$ and $W_3 = \sum_i z_i = z_2 - \frac{1}{4} \geq 0$: a bilaterally-bounded solution exists even if $z_2 + a - 1 < 0$ provided $z_2 - \frac{1}{4} \geq 0$. ■

4.2 Bi-proportional rule

This section aims at approaching proportionality under full information when solutions are required to be liability-compatible. In that purpose, we adapt the entropy objective to measure inequality in the bilateral transfers assigned by a full allocation:

$$f(\boldsymbol{b}) = \sum_{(i,j)\in\mathcal{G}} b_{ij}\left[log\left(\frac{b_{ij}}{\ell_{ij}}\right) - 1\right]. \tag{14}$$

Writing $f(\boldsymbol{b})$ as the sum over all i of $\sum_{j,(i,j)\in\mathcal{G}} b_{ij}[log(\frac{b_{ij}}{\ell_{ij}})-1]$, the ith term reflects that the ideal reimbursements by i to its creditors are proportional. Similarly, writing $f(\boldsymbol{b})$ as the sum over all i of $\sum_{j,(j,i)\in\mathcal{G}} b_{ji}[log(\frac{b_{ji}}{\ell_{ji}}) - 1]$, $f(\boldsymbol{b})$ reflects that the ideal payments to i from its borrowers are proportional. As previously, one searches for admissible solutions minimizing the entropy objective. Consider the following program:

$\mathcal{P}_1$: minimize $f(\boldsymbol{b})$ over the tight and liability-compatible solutions $\boldsymbol{b}$ of π.

Denote by $\mathcal{T}^*$ the set of problems that satisfy each inequality in (11) strictly. These problems are the ones admitting a tight and liability-compatible solution that is furthermore strictly positive on $\mathcal{G}$, hence satisfies $b_{ij} > 0$ if and only if $\ell_{ij} > 0$.

Proposition 5 *Let π be in $\mathcal{T}^*$. The solution to $\mathcal{P}_1$ is the unique super-tight solution $\boldsymbol{b}$ for which there are positive scalars $(\delta_i, \mu_i)_{i=1,\cdots,n}$ that satisfy*

$$\textit{for each } (i,j) : b_{ij} = \delta_i \ell_{ij} \mu_j \qquad \textit{(bi-proportionality)} \tag{15}$$

$$\mu_j \geq 1 \textit{ with } \mu_j = 1 \textit{ if } W_j > 0 \qquad \textit{(rescue conditions)} \tag{16}$$

Call $\boldsymbol{b}$ the constrained bi-proportional (CbiP) solution.

The feasible solutions of program $\mathcal{P}_1$ are required to be tight but not super-tight. Since the CbiP-solution is super-tight, optimizing the entropy over $\mathcal{T}^*$ thus requires the worth of the distressed firms to be null (minimal rescue).

If there are no distressed firms, the CbiP-solution is the exact one, associated to δ_i and μ_i all equal to 1. As motivated below, let us call the positive scales (μ_i) *rescue* indices (as for the CP-solution) and the positive scales (δ_i) *Reimbursement ability* (R-ability) indices. According to (15), $\boldsymbol{b}$ is obtained from $\boldsymbol{\ell}$ by multiplying i's row by R-ability index δ_i and j's column by rescue index μ_j (hence $\boldsymbol{b}$ is liability-compatible and positive on $\mathcal{G}$): $\boldsymbol{b}$ is said to be *bi-proportional* to $\boldsymbol{\ell}$.[18] The indices are defined up to a scaler: all $(c\boldsymbol{\delta}, \frac{1}{c}\boldsymbol{\mu})$ for positive c produce the same $\boldsymbol{b}$; it follows that setting the minimum of the μ_i equal to 1 is a normalization. Conditions (16) on $\boldsymbol{\mu}$ thus require the payments to a firm with positive net worth not to be upgraded relative to others. As a result, j with positive W_j receives the minimal payment per unit of claim from each of its creditors, δ_i from creditor i. In contrast, firm j with a rescue index μ_j strictly larger than 1 would be bankrupt if each other firm i reimbursed j at its minimum ratio δ_i. Consider now the scales $\boldsymbol{\delta}$. Summing over j equations (15) for a fixed i yields $b_{iN} = \delta_i \widehat{\ell}_{iN}$ where $\widehat{\ell}_{iN} = \sum_j \ell_{ij}\mu_j$ represents the sum of i's liabilities each upgraded by the corresponding rescue index. Hence, writing $\delta_i = \frac{b_{iN}}{\ell_{iN}} \frac{\ell_{iN}}{\widehat{\ell}_{iN}}$, δ_i is decomposed into the product of two factors, each less than 1. The first factor, $\frac{b_{iN}}{\ell_{iN}}$, is i's overall reimbursement ratio; it is not greater than 1 due to the upper-bound on

[18]Bi-proportional matrices appear in various areas: in statistics for adjusting contingencies tables, in economics for balancing international trade accounts (Bacharach 1965), or in voting problems (Balinski and Demange 1989-a).

i's reimbursements, and equal to 1 if i is non-distressed. The second factor, $\frac{\ell_{iN}}{\widehat{\ell}_{iN}}$, is the ratio of i's liabilities to the upgraded ones.[19] Overall δ_i is affected by i's fragility and the rescue of its creditors.

Let us examine the differences between the CP and the CbiP-solutions. When there are no distressed firms they both coincide with the exact solution. When there are distressed firms, as follows from (7), the CP-solution is defined as the CbiP-solution except that the reimbursement scale δ is common to all institutions (hence the rescue indices may differ across the two solutions; we do not introduce a different notation for simplification). This has implication on the payments to non-distressed firms (which are the only flexible quantities at a super tight solution). At the CbiP-solution, summing over i equations (15) yields $b_{Nj} = (\sum_i \delta_i \ell_{ij})\mu_j$. Thus j's payments depend on j's borrowers identities through their R-ability indices δ_i. At the CP-solution instead, payment to j is equal to $\delta \ell_{Nj}$ hence j's payments do not depend on j's borrowers identities. The same implications follow for the worth levels of non-distressed firms since they are equal to $z_j + b_{Nj} - \ell_{jN}$. To illustrate, consider two non-distressed firms whose external values, total claims and total liabilities are identical. They receive identical payments at the CP-solution but typically not at the CbiP-solution when the composition of their claims differs.

Axiomatization Clearly, the CbiP-rule applied to simple claims problems coincides with the cp-rule. The rules are more generally related through a 'consistency' principle, sometimes described as 'a part of a fair solution must be fair'. We apply this principle at a solution $\boldsymbol{b}$ to evaluate the proportionality of the reimbursements made by each single firm to the other firms, i.e. to evaluate for each i the proportionality of $(b_{ij})_{j\neq i}$. For that, imagine all reimbursements made by the firms other than i fixed to those recommended by $\boldsymbol{b}$. We are left with a simple claims problem in which b_{ij} has to be allocated to $N-i$ knowing for each j its claim on i, ℓ_{ij}, and the minimum amount j must receive from i, m^i_j, to avoid bankruptcy. Since j reimburses b_{jN} and receives $b_{N-i,j}$ from firms other than i, $m^i_j = \max(b_{jN} - z_j - b_{N-i,j}, 0)$. cp-consistency

[19] According to this computation, the CbiP-solution can be expressed in terms of rescue indices $\boldsymbol{\mu}$ and reimbursement ratios $\boldsymbol{\tau} = (\frac{b_{iN}}{\ell_{iN}})$: setting $\tau_i = \delta_i \frac{\widehat{\ell}_{iN}}{\ell_{iN}}$ we have $b_{ij} = \tau_i \mu_j \ell_{ij} \frac{\ell_{iN}}{\widehat{\ell}_{iN}}$ for each (i,j). The rescue conditions are unchanged and the τ_i, not larger than 1, are equal to 1 on D^c. Such a formulation does not easily extend to the case of bilaterally-bounded solutions studied next.

requires i's reimbursements at $\boldsymbol{b}$ to be allocated according to the cp-rule in this simple problem.

cp-consistency *Solution $\boldsymbol{b}$ is cp-consistent if, for each i, i's reimbursements $(b_{ij})_{j \neq i}$, is the cp-solution to the simple claims problem $(N-i, b_{iN}, (\ell_{ij})_{j\neq i}, (m_j^i)_{j\neq i})$.*

The next property compares the reimbursement ratios (R-ratios) of different firms. Proportionality in R-ratios requires that if i reimburses twenty percent more per unit than j a common creditor, then i reimburses twenty percent more per unit than j any other common creditor. Formally

Proportionality in R-ratios: *For each pair i and j, the reimbursement ratios to their common creditors are proportional:*

$$\frac{b_{ik}}{\ell_{ik}} / \frac{b_{jk}}{\ell_{jk}} = \frac{b_{il}}{\ell_{il}} / \frac{b_{jl}}{\ell_{jl}} \text{ for } k \text{ and } l \text{ such that } (i,k),(j,k),(i,l) \text{ and } (j,l) \text{ are in } \mathcal{G}. \tag{17}$$

Proportionality in R-ratios thus ranks debtors with common creditors. It implies that creditors with common debtors are also ranked, since (17) can be rewritten as

$$\frac{b_{ik}}{\ell_{ik}} / \frac{b_{il}}{\ell_{il}} = \frac{b_{jk}}{\ell_{jk}} / \frac{b_{jl}}{\ell_{jl}} \text{ for } k \text{ and } l \text{ such that } (i,k),(j,k),(i,l),(j,l) \text{ are in } \mathcal{G}.$$

This reads: if k is reimbursed 20% more per unit than l by a common borrower i, then k is reimbursed 20% more per unit than l by any of their common borrowers.

Proportionality in R-ratios differs from cp-consistency, because the latter bears on the reimbursements made by a *single* firm whereas the former compares reimbursements by distinct firms. Clearly, the CbiP-solution satisfies cp-consistency and Proportionality in R-ratios. The next proposition states the converse property for problems satisfying an additional condition. A *safe universal creditor* at a solution is a firm that is creditor to each other indebted firm and whose net worth is strictly positive. Such a creditor surely exists in the following problems: (a) the network is complete and $\sum_i z_i > 0$, since then there is surely a firm whose net worth is positive (b) there is a firm whose net external value covers its liabilities: say $z_1 \geq \ell_{1N}$, and who is creditor to each indebted firm: $\ell_{1i} > 0$ for each i such that $\ell_{Ni} > 0$; (b) holds in a bipartite network when a long firm is creditor to every short one and has a positive net external value.

Proposition 6 *Let π be in $\mathcal{T}^*$ and has a safe universal creditor. The CbiP-solution solution is the unique liability-compatible and super-tight solution that satisfies cp-consistency and proportionality in R-ratios.*

The existence of a safe universal creditor is necessary to characterize the CbiP-solution, as illustrated by the next example.

Example 3 Consider a bipartite network with 6 firms where 1 and 2 are short and 3, 4, 5, 6 are long: $\ell_{13} = \ell_{14} = \ell_{23} = \ell_{24} = 1$, $\ell_{15} = \ell_{26} = 2$ and all other liabilities are null. Let $z_1 = z_2 = 3$, $z_3 = z_4 = -1.7$, $z_5 = z_6 > 0$. 5 and 6 are safe so that each indebted firm, 1 or 2, has a safe creditor, but not a common one. The CbiP-solution[20] is $b_{13} = b_{23} = b_{14} = b_{24} = \frac{1.7}{2}$ and $b_{15} = b_{26} = 1.3$ with $\delta_1 = \delta_2 = \frac{1.3}{2}$, $\mu_3 = \mu_4 = \frac{1.7}{1.3}$, $\mu_5 = \mu_6 = 1$.

Let us consider $\boldsymbol{b}$ defined by $b_{13} = b_{14} = 0.9, b_{1,5} = 1.2$ and $b_{23} = b_{24} = 0.8, b_{2,6} = 1.4$, all other values null. $\boldsymbol{b}$ satisfies all conditions stated in Proposition 6. It is liability-compatible and super-tight. It satisfies Proportionality in R-ratios: 1 and 2's R-ratios to their common creditors 3 and 4 satisfy $\frac{b_{13}}{\ell_{13}} / \frac{b_{23}}{\ell_{23}} = \frac{b_{14}}{\ell_{14}} / \frac{b_{24}}{\ell_{24}} = \frac{9}{8}$. Finally $\boldsymbol{b}$ is cp-consistent. Consider first 1's reimbursements. Given the amount 0.8 reimbursed by 2, firms 3 and 4 must each receive at least 0.9 from 1; it follows that 1's reimbursements $b_{13} = b_{14} = 0.9, b_{15} = 1.2$ constitute the cp-solution supported by $\delta_1 = 0.6$ and rescue indices $\mu_3^1 = \mu_4^1 = \frac{3}{2}$. Similarly, given the amount 0,9 reimbursed by 1 to firms 3 and 4, 2's reimbursements $b_{23} = b_{24} = 0.8, b_{26} = 1.4$, constitute the cp-solution supported by $\delta_2 = 0.7$ and rescue indices $\mu_3^2 = \mu_4^2 = \frac{8}{7}$. $\boldsymbol{b}$ is not the CbiP-solution because the reimbursements by 1 and 2 are supported by different rescue indices.

A range of similar solutions satisfying all the conditions can be build: let $b_{13} = b_{14} = b$, $b_{15} = 3 - 2b$ and $b_{23} = b_{24} = 1.7 - b$, $b_{26} = 2b - 0.4$ where b is between 0.2 and 1.5. For these values, the payment by 2 to 6, $2b - 0.4$, determines the constraints faced by 1 on its reimbursements to 3 and 4; similarly the payment by 1 to 5, $3 - 2b$, determines the constraints faced by 2 on its reimbursements to 3 and 4. As 5 and 6 have no common debtors, the payments they receive respectively from 1 and 2 are

[20] Surely $\mu_5 = \mu_6 = 1$ so that reimbursements made by $i = 1, 2$ satisfy: $b_{i3} = \delta_i \mu_3$, $b_{i4} = \delta_i \mu_4$, $b_{i5} = 2\delta_i$ and $b_{i6} = 2\delta_i$ with a total $\delta_i(\mu_3 + \mu_4 + 2)$ summing to 3. This implies $\delta_1 = \delta_2 = \delta$, hence $b_{13} = b_{23} = \delta\mu_3$ and $b_{14} = b_{24} = \delta\mu_4$. Since W_3 and W_4 are null, we must have $-1.7 + 2\delta\mu_3 = 0$ and $-1.7 + 2\delta\mu_4 = 0$. We thus obtain $\mu_3 = \mu_4 = \mu$ and $\delta\mu = 1.7/2$. Plugging this value into 1 and 2 's total reimbursement, $\delta(2\mu + 2) = 3$, yields the stated solution.

independent, explaining the indetermination in b. Assume instead that 5 and 6 have each lent $\frac{3}{4}$ to 1 and to 2. The solution $\boldsymbol{b}$ where each receives half of $3-2b$ from 1 and half of $2b-0.4$ from 2, keeping everything else unchanged, is cp-consistent. It does not satisfy Proportionality in R-ratios except if $3-2b=2b-0.4$, i.e. $b=\frac{1.7}{2}$ corresponding to the CbiP-solution. ∎

4.3 Bilaterally-bounded bi-proportional rule

This section aims at defining proportionality when solutions are constrained to be bilaterally-bounded. We consider program $\mathcal{P}_2$, which has the same entropy objective as $\mathcal{P}_1$ but requires solutions to be bilaterally-bounded instead of liability-compatible:

$$\mathcal{P}_2: \text{ minimize } f(\boldsymbol{b}) \text{ over the tight and bilaterally-bounded solutions}$$

where f is defined by (14). Let $\mathcal{T}_b^*$ denote the set of problems that satisfy each inequality in (13) strictly. These problems are the ones admitting tight and bilaterally-bounded solutions that are positive for each (i,j) in $\mathcal{G}_D$.

Proposition 7 *Let π be in $\mathcal{T}_b^*$. The solution to $\mathcal{P}_2$ is the unique super-tight solution $\boldsymbol{b}$ such that, for scales $(\delta_i,\mu_i), i=1,\cdots,n$, all positive:*

$$\textit{for each } (i,j): b_{ij}=\min(\delta_i\mu_j,1)\ell_{ij} \tag{18}$$

with $\boldsymbol{\mu}$ satisfying the rescue conditions (16). Call $\boldsymbol{b}$ the constrained bounded bi-proportional solution (CbbiP).

According to (18), reimbursements are described by a 'capped' version of a matrix bi-proportional to $\boldsymbol{\ell}$: after multiplying i's liabilities by δ_i and j's claims by μ_j, the transfer is capped to ℓ_{ij} if necessary. μ_j still satisfies the rescue condition according to which its claims are scaled up only to avoid its bankruptcy.

Next proposition provides an axiomatization of the CbbiP-solution based on the same ideas underlying cp-consistency and proportionality in R-ratios. Definitions are adapted to account for the bilateral bounds. From Property 2, a bilaterally-bounded and tight solution is fixed for the pairs not in $\mathcal{G}_D$: cp-consistency thus applies only to its restriction on $\mathcal{G}_D$.

cp-consistency on $\mathcal{G}_D$: *Bilaterally-bounded and tight solution $\boldsymbol{b}=(\boldsymbol{b}_{|\mathcal{G}_D},\boldsymbol{\ell}_{|N^2-\mathcal{G}_D})$ is cp-consistent on $\mathcal{G}_D$ if $\boldsymbol{b}_{|\mathcal{G}_D}$ is cp-consistent.*

Constrained proportionality in R-ratios accounts for the bilateral bounds by requiring that the reimbursement ratios of two firms to their common creditors are proportional except if this would violate the bilateral bounds:

Constrained proportionality in R-ratios: *For each i, j, k and l such that (i,k), (j,k), (i,l), and (j,l) are in $\mathcal{G}$, $\frac{b_{ik}}{\ell_{ik}}/\frac{b_{jk}}{\ell_{jk}} < \frac{b_{il}}{\ell_{il}}/\frac{b_{jl}}{\ell_{jl}}$ implies $b_{ik} = \ell_{ik}$ or $b_{jl} = \ell_{jl}$.*

Proposition 8 *Let π be in $\mathcal{T}_b^*$ and has a safe universal creditor. The constrained bounded bi-proportional solution is the unique bilaterally-bounded and super-tight solution that satisfies cp-consistency on $\mathcal{G}_D$ and constrained proportionality in R-ratios.*

We illustrate the differences between the CbiP and CbbiP-solutions in Example 1.

Example 1-d In Example 1, one easily checks that the CbiP and CbbiP-solutions coincide for z_2 larger than $\frac{1}{1+a}$. All rescue indices are equal to 1 (even that of firm 1), so that 1's reimbursements are proportional to its liabilities: $b_{12} = \frac{a}{1+a}$ and $b_{13} = \frac{a^2}{1+a}$ with $\delta_1 = \frac{a}{1+a}$ and 2 and 3 reimburse fully each of their liabilities, $\delta_2 = \delta_3 = 1$. It follows that $W_2 = z_2 - \frac{1}{1+a}$. For z_2 smaller than $\frac{1}{1+a}$, 2 must be repaid more in proportion than 3 to avoid its bankruptcy so that 2's rescue index is thus larger than 1. Hence 2's worth must be null so that 2 receives in total $(1+a)-z_2$. The CbiP and CbbiP-solutions differ to achieve this total. At the CbbiP-solution, denoted by bb, 3's reimbursement to 2 is fixed equal to a, hence any decrease in z_2 must be compensated by an increase in 1's reimbursement to 2, implying $bb_{12} = 1-z_2$, hence $bb_{13} = a-1+z_2$. The solution exists provided bb_{13} and W_3 are non-negative, i.e. for $z_2 \geq 1-a$ and $z_2 \geq \frac{1}{4}$. At the CbiP-solution, 3's reimbursement to 2 is not capped by liability a, so that a decrease in z_2 induces an increase in b_{32}, which in turn implies that 1's reimbursement to 2 increases less than under the CbbiP-solution. Thus $bb_{13} \leq b_{13}$ and bb_{12} is larger and steeper than b_{12}, as displayed in the left Figure 1, drawn for $a = 0.6$. The CbiP-solution is defined for $z_2 \geq \frac{1}{4}$ and the CbiP-ones for $z_2 \geq 0.4$. Requiring bilateral-boundedness instead of liability-compatibility thus restricts the possibility for tight solutions to exist.

Consider now 2 and 3's worths at the CP, CbiP and CbbiP-solutions. The CP-solution (computed in example 1-b) largely differs from the CbiP-solution because it ignores the bilateral liabilities and claims, which much differ across the two firms. This translates into very different worths as displayed in Figure 1 on the right. In

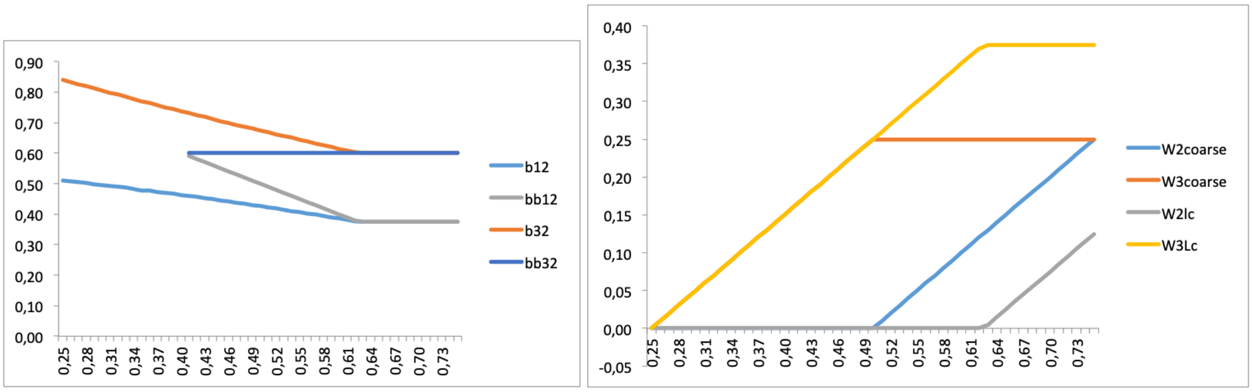

Figure 1: Transfers and net worth $a = 0.6$

this example, the worths at the CbiP and CbbiP-solutions coincide, the solutions differ only when 2's worth is null, in which case $W_3 = z_1 + z_2 + z_3$ at any solution since both W_1 and W_2 are null. This is specific to this example because transfers are very constrained.[21] To see this, let us modify the example by splitting firm 3 into two firms, $3'$ and $3''$, as follows by dividing equally 3's net values and total liabilities and claims but making firm 1 indebted to 3' only and firm 2 indebted to $3''$ only: $z_{3'} = z_{3''} = 3/8$ and the liabilities matrix is

$$\boldsymbol{\ell} = \begin{pmatrix} 0 & 1 & a & 0 \\ a & 0 & 0 & 1 \\ \frac{1}{2} & a/2 & 0 & 0 \\ \frac{1}{2} & a/2 & 0 & 0 \end{pmatrix}$$

Using the same argument as above, the CbiP-and CbbiP-solutions differ only when W_2 is null, which arises for z_2 less than $\frac{1}{1+a}$. Assume this is the case. $3''$ is creditor of non-distressed 2 only, hence is fully repaid at the CbbiP-solution, which yields the worth $W_{3''} = z_{3''} + 1 - \frac{1+a}{2}$, or equal to $\frac{7}{8} - \frac{a}{2}$, independently of he value of z_2. Consider now the CbiP-solution. Since $3'$ and $3''$ are indebted to 2, both increase their reimbursements to 2 above their liabilities when z_2 decreases below $\frac{1}{1+a}$; this implies that $3''$'s worth decreases with z_2, hence is strictly lower than $W_{3''}$ (the reverse inequality holds for $3'$ since $3'$ and $3''$ worths sum up to $z_2 - \frac{1}{4}$). ■

Computation by hand is usually not possible. Algorithms for finding the matrix

[21]The smaller reimbursement from 1 to 2 at the CbbiP-solution is exactly compensated by a larger reimbursement from 1 to 3, itself compensated by a larger reimbursement from 3 to 2, i.e. $bb_{12} - b_{12} = b_{13} - bb_{13} = b_{32} - bb_{32}$, leaving worths unchanged.

with specified totals rows and columns bi-proportional to another one have been studied extensively (see e.g. Balinski and Demange 1989-b and Censor and Zenios 1997 survey). These algorithms, based on iteratively scaling the rows and the columns, can be adapted to take into account the bankruptcy constraints and the bilateral bounds. Performing simulations with 20 institutions and positive liabilities, convergence is fast though we have not studied analytically the speed.

4.4 Concluding remarks

The paper examines how to apply the proportionality principle to the resolution of cross-liabilities in a system, when resolution must in priority avoid default to external creditors and, as far as possible, limit the discrepancy from exact reimbursements. Two broad classes of constrained-proportional rules are considered: Coarse rules, which do not take into account bilateral liabilities, in particular who has lent to distressed firms and which amount, and full rules which do and which specify bilateral transfers related to the known bilateral liabilities. Coarse rules facilitate resolution but may implement transfers that are illegal or not accepted if the bilateral liabilities are known. Full rules instead account for this knowledge, which strongly affects bilateral reimbursements: (constrained) proportionality requires a differential treatment of creditors according to their health and to the need to rescue them. As a result, the net worth of the firms depend not only on the composition of their claims but also of their liabilities.

The analysis could be developed in several directions. So far, it considers positions at a resolution stage, without addressing the interaction between the chosen resolution rule and the incentives for a firm to lend and borrow or to enter into contracts at an earlier stage. In practice, firms have incentives to screen debtors and extend loans consequently since debtor's reimbursements depend on their health (although only partially due to limited liability). But with cross-liabilities, propagation of defaults arises from borrowers to their creditors, then to the creditors' creditors, and so on. The incentives to screen creditors and borrow from safe ones thus should also be enhanced to improve the robustness of the system: a rule that accounts for these incentives is less prone to favor contagion. In this respect, when comparing the constrained-proportional rules we have defined, the full ones, which specify larger reimbursements to rescued creditors, provide better incentives to screen creditors than the coarse ones. Studying such incentives in general resolution rules, not necessarily proportional ones, is worth further investigation.

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5 Proofs

Proof of Property 1 Let the nominal worth of each firm be positive. Define the set $A = \{i, b_{iN} = \ell_{iN}\}$. Reimbursement-boundedness implies $A^c = \{i, b_{iN} < \ell_{iN}\}$. Assume, by contradiction, A^c non-empty. Creditor's priority implies that for each i in A^c: $W_i = 0$, hence summing over A^c: $W_{A^c} = 0$. We find a contradiction with the positivity of their nominal worth by showing

$$z_{A^c} + b_{N,A^c} - b_{A^c,N} \geq z_{A^c} + \ell_{N,A^c} - \ell_{A^c,N} \tag{19}$$

The left hand side is W_{A^c}, which is null, and the right hand side is the sum of the nominal worth of firms in A^c, which are positive: a contradiction. Let us show (19). Transfers are balanced so that $b_{N,A^c} - b_{A^c,N} = b_{A,N} - b_{N,A}$. By definition of A, $b_{iN} = \ell_{iN}$ for i in A, which implies $b_{A,N} = \ell_{A,N}$; by payment boundedness $b_{N,A} \leq \ell_{N,A}$. Hence $b_{N,A^c} - b_{A^c,N} \geq \ell_{A,N} - \ell_{N,A}$. Since $\ell_{A,N} - \ell_{N,A} = \ell_{N,A^c} - \ell_{A^c,N}$, this proves $b_{N,A^c} - b_{A^c,N} \geq \ell_{N,A^c} - \ell_{A^c,N}$, hence (19). Since A^c is empty, $b_{iN} = \ell_{iN}$ for each i so that $b_{NN} = \ell_{NN}$, which in turn implies $b_{Ni} = \ell_{Ni}$ for each i by payment-boundedness. ∎

Proof of Proposition 2. Let us write down program $\mathcal{P}$:

$$\begin{aligned}
\mathcal{P}: \min f(\boldsymbol{B}) = \quad & a\sum_i b_{iN}[log(\frac{b_{iN}}{\ell_{iN}})-1]+(1-a)\sum_i b_{Ni}[log(\frac{b_{Ni}}{\ell_{Ni}})-1] \\
\text{over} \quad & \boldsymbol{B}=(b_{iN},b_{Ni})_{i=1,\cdots,n}\geq 0 \text{ satisfying the constraints :} \\
\text{for each } i \quad : \quad & b_{iN}-b_{Ni}\leq z_i,\ b_{iN}\leq \ell_{iN} \text{ and } b_{Ni}\leq \ell_{Ni} & (20) \\
\text{and} \quad & \sum_i b_{Ni}=\sum_i b_{iN} & (21)
\end{aligned}$$

The inequalities (20) state for each firm the non-negativity of its worth and the bounds on its reimbursements and payments. Equation (21) states the balancedness condition. The objective function f of $\mathcal{P}$ is separable and strictly convex because x log$x$ is strictly convex, with a global minimum equal to -1 reached at $x=1$. The feasible set of $\mathcal{P}$, defined by linear inequalities, has a nonempty interior for Π in $\mathcal{C}^*$. It follows that the solution is unique, characterized by the first order conditions on the Lagrangian. We show that the unique solution to $\mathcal{P}$ coincides with the CP-solution. This is obvious if no firm is distressed since then the CP-solution is the exact solution, $b_{iN}=\ell_{iN}$ and $b_{Ni}=\ell_{Ni}$ for each i, which is also the unique solution to $\mathcal{P}$ because it is feasible and produces the global minimum $-\ell_{NN}$ of f.

Assume now there are distressed firms: $D\neq\emptyset$. Denote respectively by α_i, β_i and γ_i the non-negative Kuhn-Tucker multipliers associated to i's constraints on net worth and reimbursements and payments and by λ the multiplier on the balance condition. The Lagrangian $\mathcal{L}$ is equal to

$$f(\boldsymbol{B})+\sum_i[\alpha_i(b_{iN}-b_{Ni}-z_i)+\beta_i(b_{iN}-\ell_{iN})+\gamma_i(b_{Ni}-\ell_{Ni})+\lambda(b_{Ni}-b_{iN})]$$

where $\alpha_i\geq 0,\beta_i\geq 0,\gamma_i\geq 0$. The first order and complementarity conditions are for each i:

$$\frac{\partial\mathcal{L}}{\partial b_{iN}}=a\ \log\frac{b_{iN}}{\ell_{iN}}+\alpha_i+\beta_i-\lambda\leq 0 \text{ with } = \text{ if } b_{iN}<\ell_{iN} \tag{22}$$

$$\frac{\partial\mathcal{L}}{\partial b_{Ni}}=(1-a)\ \log\frac{b_{Ni}}{\ell_{Ni}}-\alpha_i+\gamma_i+\lambda\leq 0 \text{ with } = \text{ if } b_{Ni}<\ell_{Ni}. \tag{23}$$

$$\alpha_i(b_{iN}-b_{Ni}-z_i)=0,\ \beta_i(b_{iN}-\ell_{iN})=0,\ \gamma_i(b_{Ni}-\ell_{Ni})=0 \tag{24}$$

Claim 1: $b_{Ni}<\ell_{Ni}$ if and only if $\lambda>\alpha_i$, in which case $b_{Ni}=\exp(\frac{\alpha_i-\lambda}{1-a})\ell_{Ni}$.

Proof: Assume $b_{Ni} = \ell_{Ni}$. (23) implies $-\alpha_i + \gamma_i + \lambda \leq 0$ hence a fortiori $-\alpha_i + \lambda \leq 0$ since $\gamma_i \geq 0$. This proves the if part. Assume $b_{Ni} < \ell_{Ni}$. Then $\gamma_i = 0$ by (24) and $\frac{\partial \mathcal{L}}{\partial b_{Ni}} = 0$ by (23). These two conditions imply $(1-a)\log\frac{b_{Ni}}{\ell_{Ni}} - \alpha_i + \lambda = 0$, which requires $-\alpha_i + \lambda > 0$ and yields the stated value for b_{Ni}. This proves the only if part.

Claim 2: $\lambda > 0$. Proof: By contradiction, assume $\lambda = 0$. Since the α_i are non-negative, Claim 1 implies that each i is fully repaid: $b_{Ni} = \ell_{Ni}$, which in turn implies that each firm fully reimburses its liabilities (since $\boldsymbol{B}$ is $R\&P$-bounded and balanced, we can use the same argument as at the end of the proof of Property 1). Hence the (non-negative) net worth of each firm is equal to its nominal one: no firm is distressed, the desired contradiction.

Claim 3: $b_{iN} < \ell_{iN}$ if and only if $\lambda < \alpha_i$, in which case $b_{iN} = \exp(\frac{-\alpha_i+\lambda}{a})\ell_{iN}$.

Proof: Assume $b_{iN} < \ell_{iN}$. Then $\beta_i = 0$ by the complementarity condition (24) and $\frac{\partial \mathcal{L}}{\partial b_{iN}} = 0$ by (22). These two conditions imply $a\log\frac{b_{iN}}{\ell_{iN}} + \alpha_i - \lambda = 0$, which implies $\alpha_i - \lambda > 0$ and the stated value for b_{iN}. This proves the only if part. To prove the converse, assume $b_{iN} = \ell_{iN}$. (22) writes $\alpha_i + \beta_i - \lambda \leq 0$ hence surely $\alpha_i - \lambda \leq 0$.

Claim 4: $\boldsymbol{B}$ is super-tight. Furthermore, for i distressed: $\lambda < \alpha_i$ and for i non-distressed: $\lambda \geq \alpha_i$ and $b_{Ni} = \exp(\frac{\alpha_i-\lambda}{1-a})\ell_{Ni}$.

Proof: Claims 1 and 3 prove that the following three cases are possible for i: (i) $b_{Ni} = \ell_{Ni}$, $b_{iN} < \ell_{iN}$ and $\lambda < \alpha_i$ (ii) $b_{Ni} = \ell_{Ni}$, $b_{iN} = \ell_{iN}$ and $\lambda = \alpha_i$ (iii) $b_{Ni} < \ell_{Ni}$, $b_{iN} = \ell_{iN}$ and $\lambda > \alpha_i$. i's nominal net worth, $z_i + \ell_{Ni} - \ell_{iN}$, is negative in case (i), null in case (ii) and positive in case (iii). A distressed firm is thus in case (i). It is fully repaid and furthermore its net worth is null because $\alpha_i > 0$ (since λ is positive by Claim 2): minimal rescue holds. The non-distressed firms fully reimburse their liabilities since they are in case (ii) or (iii). This proves that $\boldsymbol{B}$ is super-tight. Finally, in case (iii), $b_{Ni} = \exp(\frac{\alpha_i-\lambda}{1-a})\ell_{Ni}$ by Claim 1. The same expression holds in case (ii) since $\lambda = \alpha_i$ and $b_{Ni} = \ell_{Ni}$.

End of the Proof: It remains to prove (7). From Claim 4, $b_{Ni} = \exp(\frac{\alpha_i-\lambda}{1-a})\ell_{Ni}$ for $i \in D^c$. Thus $b_{Ni} = \delta\mu_i\ell_{Ni}$ by defining $\delta = \exp(\frac{-\lambda}{1-a})$ and $\mu_i = \exp(\frac{\alpha_i}{1-a})$. The rescue condition is met since $\alpha_i \geq 0$ and $\alpha_i > 0$ only if W_i is null. This proves (7). ∎

Proof of Proposition 3. Consider a rule F defined on $\mathcal{C}$, that assigns a super-tight solutions, satisfies proportional payments' target. and worths' monotony with respect to $\boldsymbol{z}$. Let Π^0 be a problem in $\mathcal{C}$ with net external values z^0 and distressed set D. Defining $\underline{z}_i = \ell_{iN} - \ell_{Ni}$, we have $z_i^0 < \underline{z}_i$ for each i in D and $z_i^0 \geq \underline{z}_i$ for i in D^c.

Consider problems Π with a net external value z_i in $\in [z_i^0, \underline{z}_i]$ for each i in D, keeping the net external values of the firms in D^c and the claims and liabilities ℓ unchanged. For $z_D = \underline{z}_D$, the distressed set is empty. Since F is exact, $F(\Pi) = (\ell_{iN}, \ell_{Ni})_{i=1,\cdots,n}$ so that i's net worth is null: $W_i(\Pi) = 0$.

Consider now problems with $z_i \in [z_i^0, \underline{z}_i[$ for each i in D. All belong to $\mathcal{C}$ and have the same distressed set D. We show that F assigns fixed payments to firms in D. For $z_D < \underline{z}_D$, i's net worth is not greater than 0 by worth monotony with respect to $\boldsymbol{z}$, hence it is null: $W_i(\Pi) = 0$. Since i is fully repaid, $b_{Ni}(\Pi) = \ell_{Ni}$, we must have $b_{iN}(\Pi) = z_i + \ell_{Ni}$ (which is non-negative under (5)): reimbursements by firms D are constant. To determine the payments to the non-distressed firms, we apply Proposition 1. The payments $(b_{Ni})_{i \in D^c}$ to D^c sum to $T = \ell_{N,D^c} + z_D$ and are allocated according to a rule f on simple claims problems. We show that f satisfies the assumptions of Lemma 1 below. First, Proportional payments' target is straightforward. Second, f is monotone with respect to the total T due to the monotony of net worth levels with respect to $\boldsymbol{z}$ and the fact that the total T is increasing in z_D. Applying Lemma 1, the payments $(b_{Ni}(\Pi))_{i \in D^c}$ are given by the cp-rule, hence satisfy (7). ∎

Lemma 1 *Let $\mathcal{B}$ be the set of simple claims problems, $\sigma = (J, T, \mathbf{c}, \mathbf{m})$, $c_J \leq T$, satisfying $m_J \leq T$. The cp-rule on $\mathcal{B}$ is the unique rule that satisfies Proportional payment's target and Monotony with respect to the estate: $T < T'$ implies $x_j \leq x'_j$ where $\boldsymbol{x} = F(J, T, \mathbf{c}, \mathbf{m})$ and $\boldsymbol{x}' = F(J, T', \mathbf{c}, \mathbf{m})$.*

Proof. Let $\sigma = (J, T^0, \mathbf{c}, \mathbf{m})$ be in $\mathcal{B}$. We consider problems with values for T larger than T^0, keeping $\mathbf{c}, \mathbf{m}$ fixed. Define $\delta_j = \frac{m_j}{c_j}$. $\delta_j c_j$ is the minimum amount that j must receive. Order the distinct values of the δ_j by decreasing order: $\delta^{(1)} > \delta^{(2)} > \cdots > \delta^{(k)} \cdots > \delta^{(K)}$.

Define $T^1 = \min(\delta^{(1)}, 1)c_J$. For T larger than T^1, the proportional allocation of T to J satisfies the minimum constraints, so proportional payments target implies it is the solution. In particular, at $T = T^1$, each j receives $\delta^{(1)} c_j$; thus, $x_j = m_j$ for each j in J^1 where J^1 denotes the set of j for which $\delta_j = \delta^{(1)}$.

Let $T \leq T^1$. By monotony with respect to the estate, the payments to J^1 are not larger than those received for $T = T^1$, which are already minimal. This implies that for any $T \leq T^1$ each j in J^1 receives $m_j = \delta^{(1)} c_j$. It remains to find the allocation to $J - J^1$ of what is left $T - \delta^{(1)} c_{J^1}$. We use the same argument as above starting from T^1. Note first that at $T = T^1 = \delta^{(1)}c_{J^1}$, the amount allocated to $J - J^1$ is $T^1 - \delta^{(1)}c_{J^1} = \delta^{(1)}c_{J-J^1}$, which is strictly larger than $\delta^{(2)}c_{J-J^1}$ by definition of the $\delta^{(k)}$. Thus, there is T^2 such that for T in the interval (T^2, T^1) the inequality $T - \delta^{(1)}c_{J^1} > \delta^{(2)}c_{J-J^1}$ holds ($T^2 = \delta^{(1)}c_{J^1} + \delta^{(2)}c_{J-J^1}$). For such T allocating $T - \delta^{(1)}c_{J^1}$ to $J - J^1$ in proportion of their claims is the solution. If $T^0 \geq T^2$ we are done: the solution is the cp-solution associated to $\delta = \delta^{(2)}$, $\mu_j = \delta^{(1)}/\delta^{(2)}$ for j in J^1 and $\mu_j = 1$ for j in $J - J^1$. Otherwise, $T^0 < T^2$. Repeating the argument from T^2 until reaching T^0 shows that firms receive the cp-solution. ∎

Proof of Proposition 4. For each of the four cases considered in the proposition, we define a graph with lower and upper capacities for which the existence of a circulation is equivalent to the existence of a solution satisfying the required constraints. Then we use Hoffmann theorem, which characterizes the graphs for which a circulation exists. Let $d(e)$ and $u(e)$ denote the lower and upper capacity of edge e. For a nonempty subset X of the node set V, define the upper capacity of its outgoing edges by $u_-(X) = \sum_{e=(i,j),i\in X,j\notin X} u(e)$ and lower capacity of its ingoing edges by $d_+(X) = \sum_{e=(i,j),i\notin X,j\in X} d(e)$. Hoffmann theorem states: either there exists no circulation satisfying the capacity constraints or

$$u_-(X) \geq d_+(X) \text{ for any subset } X. \tag{25}$$

Let us start with bilaterally-bounded solutions, simpler to handle than liability-compatible ones.

Conditions (13) Consider the directed network with node set $V = \{0\} \cup N = \{0, \cdots, n\}$ and the following edges and capacities:

$(0, i)$ for $i \in N$ with $d(e) = z_i$, $u(e) = z_i$

$(i, 0)$ for $i \in N$ with $d(e) = 0$, $u(e) = \infty$

(i, j) for each i and j both in N with $d(e) = 0$ if (i, j) is in $\mathcal{G}_D$, and $d(e) = \ell_{ij}$ if (i, j) is not in $\mathcal{G}_D$; $u(e) = \ell_{ij}$.

It is easy to check that a circulation is associated to a tight and bilaterally-bounded solution and conversely by setting b_{ij} equal to the flow on edge (i, j) (see the detailed proof for proving (11) on liability-compatible solutions). Given X nonempty subset of V, let us compute $u_-(X)$ and $d_+(X)$. Denote $I = N \cap X$. Consider two cases.

Case 1: $0 \notin X$. Surely I is non-empty (since $X \neq \emptyset$). The edge $(i, 0)$ for $i \in I$ is outgoing from X. Since it has an infinite upper capacity, this implies $u_-(X) = \infty$

hence (25) is satisfied.

Case 2: $0 \in X$. The outgoing edges from 0 to $V - X$ are of the form $(0, i)$ for $i \notin I$, each with an upper-capacity equal to z_i; the outgoing edges from a node in N are of the form (i, j) where $i \in I$ and $j \notin I$, with an upper-capacity equal to ℓ_{ij}; we thus obtain the total upper-capacity: $u_-(X) = z_{I^c} + \ell_{I,I^c}$. The lower capacity of the ingoing edges are null except for the edges (i, j) in $\mathcal{G} - \mathcal{G}_D$ when $i \notin I$ and $j \in I$, each with a lower-capacity equal to ℓ_{ij}. The edges (i, j) not in $\mathcal{G}_D$ are those where i is not distressed and those where i and j are distressed. We thus obtain $d_+(X) = \ell_{I^c \cap D^c, I} + \ell_{I^c \cap D, I \cap D}$. Hoffman condition (25) thus writes $z_{I^c} + \ell_{I,I^c} \geq \ell_{I^c \cap D^c, I} + \ell_{I^c \cap D, I \cap D}$. This inequality must be satisfied for each subset I, hence denoting I^c by A this proves (13). ■

Conditions (12) It suffices to set the lower capacities of all edges (i, j) to 0. ■

Conditions (11). Consider the directed network with node set $V = \{0, \cdots 3n\}$ and the following edges e:

$(0, i)$, for $i \in N$ and z_i with abilities equal to z_i: $d(e) = z_i$, $u(e) = z_i$

$(i, 0)$ for $i \in N$ with $d(e) = 0$, $u(e) = \infty$

$(i, i + n)$ for $i \in N$ with $d(e) = 0$ for $i \in D$, $d(e) = \ell_{iN}$ for $i \in D^c$ and $u(e) = \ell_{iN}$

$(i + n, j + 2n)$ if (i, j) is in $\mathcal{G}$ with $d(e) = 0$, $u(e) = \infty$

$(i + 2n, i)$ for $i \in N$ with $d(e) = 0$ for $i \in D^c$, $d(e) = \ell_{Ni}$ for $i \in D$ and $u(e) = \ell_{Ni}$.

Let us check that a circulation is associated to a tight and liability-compatible solution and conversely. Let x_e denote the flow on edge e. The circulation conditions write:

$$\begin{aligned}
\text{at } i: &\quad z_i + x_{i+2n,i} &&= x_{i,i+n} + x_{i,0} && (26)\\
\text{at } i+n: &\quad x_{i,i+n} &&= \sum_{j,(i,j)\in\mathcal{G}} x_{i+n,j+2n} && (27)\\
\text{at } i+2n: &\quad \textstyle\sum_{j,(j,i)\in\mathcal{G}} x_{j+n,i+2n} &&= x_{i+2n,i} && (28)\\
\text{at } 0: &\quad \textstyle\sum_i z_i &&= \sum_i x_{i,0} && (29)
\end{aligned}$$

and the capacity constraints:

$$\begin{aligned}
&\text{for each } (i,j) \in \mathcal{G}: x_{i+n,j+2n} \geq 0, \quad &&\text{for each } i: x_{i0} \geq 0, x_{0,i} = z_i && (30)\\
&\text{for each } i \in D: 0 \leq x_{i,i+n} \leq \ell_{iN}, \quad &&\text{for each } i \in D^c: x_{i,i+n} = \ell_{iN} && (31)\\
&\text{for each } i \in D^c: 0 \leq x_{i+2n,i} \leq \ell_{Ni}, \quad &&\text{for each } i \in D: x_{i+2n,i} = \ell_{Ni} && (32)
\end{aligned}$$

We associate to circulation $\boldsymbol{x}$ the allocation $\boldsymbol{b}$ defined by b_{ij} equal to $x_{i+n,j+2n}$ for each (i, j) in $\mathcal{G}$ and 0 otherwise. $\boldsymbol{b}$ is non-negative and liability-compatible. From equation (27), $x_{i,i+n}$ is equal to i's total reimbursement b_{iN} and from (28), $x_{i+2n,i}$ is equal to i's total payment b_{Ni}. Hence equation (26) implies that $x_{i,0}$ is i's net worth, which is non-negative by (30): $\boldsymbol{b}$ is a solution. Furthermore, the capacity constraints (31) ensure that b_{iN} is bounded above by ℓ_{iN} with an equality for a non-distressed firm, and (32) on $(i + 2n, i)$ that b_{Ni} is bounded above by ℓ_{Ni} with an equality for a distressed firm: $\boldsymbol{b}$ is tight. Conversely, a tight and liability-compatible solution $\boldsymbol{b}$ easily defines a circulation. (Equation (29) simply states that the aggregate net external value is equal to the aggregate net worth.)

Given X subset of V, let us compute $u_-(X)$ and $d_+(X)$. Denote $I = \{i \in N \text{ s.t. } i \in X\}$, $J = \{i \in N \text{ s.t. } i + n \in X\}$ and $K = \{i \in N \text{ s.t. } i + 2n \in X\}$.

Case 1: $0 \in X$. Consider the outgoing edges

(i) from 0: they are of the form $(0, i)$ for $i \notin I$ with upper-capacity z_i;

(ii) from $i \in X \cap N$, i.e. $i \in I$: the unique edge from i, $(i, i + n)$, is outgoing if $i + n \notin X$, i.e. $i \notin J$. Their total upper-capacity is equal to $\ell_{I \cap J^c, N}$.

(iii) from $j + n \in X$, i.e. $j \in J$: they are of the form $(j + n, k + 2n)$ if $(j, k) \in \mathcal{G}$ and $k \notin K$; their upper-capacity is infinite

(iv) from $k+2n \in X$, i.e. $k \in K$: the unique edge from $k+2n$, $(k+2n, k)$ is outgoing if $k \notin X$, that is if $k \notin I$. Their upper-capacity is thus $\ell_{N, K \cap I^c}$

(25) is automatically satisfied for X with $u_-(X) = \infty$. From (iii), this is the case if there is an edge (j, k) in $\mathcal{G}$ with $j \in J$ and $k \notin K$, i.e. a creditor of J is not in K, equivalently a debtor of K^c is in J. We thus now restrict to X such that no debtor of K^c is in J: $D(K^c) \subset J^c$. Summing over all outgoing edges, we obtain

$$u_-(X) = z_{I^c} + \ell_{I \cap J^c, N} + \ell_{N, K \cap I^c}$$

To compute $d_+(X)$, note that the lower-capacity of an edge is null except for edge $(i, i+n)$ if $i \in D^c$ with a lower capacity equal to ℓ_{iN} or edge $(i+2n, i)$ if $i \in D$ with a lower capacity equal to ℓ_{Ni}. Since $(i, i+n)$ is an ingoing edge if $i \notin X$ and $i+n \in X$ and $(i + 2n, i)$ is ingoing if $i + 2n \notin X$ and $i \in X$, we obtain

$$d_+(X) = \ell_{D^c \cap I^c \cap J, N} + \ell_{N, D \cap K^c \cap I}.$$

So Hoffman's condition (25) writes

$$z_{I^c} + \ell_{I\cap J^c,N} + \ell_{N,K\cap I^c} \geq \ell_{D^c\cap I^c\cap J,N} + \ell_{N,D\cap K^c\cap I} \text{ where } D(K^c) \subset J^c. \tag{33}$$

Case 2: $0 \notin X$. Since an edge $(i, 0)$ has an infinite upper-capacity, we only need consider X for which there is no such outgoing edge, i.e. $I = \emptyset$. The upper-capacity of outgoing edges is determined as in Case 1 applied to $I = \emptyset$. The lower-capacity of ingoing edges to X is also given by the same expression because the edges from 0 are of the form $(0, i)$ but no one is ingoing since no i is in X. The Hoffmann's condition thus writes as (33) for $I = \emptyset$.

Given K, condition (33) is the strongest one for J^c the smallest one, i.e. $D(K^c) = J^c$, so (33) reduces to

$$z_{I^c} + \ell_{I\cap D(K^c),N} + \ell_{N,K\cap I^c} \geq \ell_{D^c\cap I^c\cap D(K^c)^c,N} + \ell_{N,D\cap K^c\cap I}.$$

Replacing I^c by A, K^c by B, we obtain (11).

Conditions (10). It suffices to set the lower capacities of all edges $(i, i+n)$ and $(i+2n, i)$ to zero. ■

Proof of Proposition 5. A liability-compatible solution is of the form $\boldsymbol{b} = (\boldsymbol{b}_{|\mathcal{G}}, \mathbf{0}_{|N^2-\mathcal{G}})$, hence the contributions of the transfers outside $\mathcal{G}$ to the entropy measure f are fixed. The program $\mathcal{P}_1$ writes

$\mathcal{P}_1 : \min f(\boldsymbol{b}_{|\mathcal{G}}) = \sum_{(i,j)\in\mathcal{G}} b_{ij} \left[log\left(\frac{b_{ij}}{\ell_{ij}}\right) - 1\right]$ over the $(\boldsymbol{b}_{|\mathcal{G}}, \mathbf{0}_{|N^2-\mathcal{G}})$ that satisfy

$$\text{for each } i \;:\; \sum_j b_{ij} - \sum_j b_{ji} \leq z_i \tag{34}$$

$$\text{for each } i \;:\; \sum_j b_{ij} \leq \ell_{iN} \text{ with an equality for } i \notin D \tag{35}$$

$$\text{for each } i \;:\; \sum_j b_{ji} \leq \ell_{Ni} \text{ with an equality for } i \in D. \tag{36}$$

For a problem in $\mathcal{T}^*$, the conditions (11) are satisfied strictly, so that the feasible set of $\mathcal{P}_1$ has a non-empty interior. The proof follows similar lines as that of Proposition 4 by showing that the solutions to convex program $\mathcal{P}_1$ coincide with the CbiP-solutions, proving both the existence and uniqueness of a CbiP-solution. This is obvious if no firm is distressed since then $b_{ij} = \ell_{ij}$ for each i, j is the unique solution to $\mathcal{P}_1$ and is the CbiP-solution. So we assume that there are distressed firms.

Consider the Lagrangian of $\mathcal{P}_1$. Denote by α_i the Kuhn-Tucker multiplier to i's net worth constraint (34), by β_i the one on i's reimbursements (35), by γ_i the one on i's payments (36). To simplify the presentation, we assume each firm has a debtor and a creditor (otherwise it suffices to drop the corresponding constraint and set $\beta_i = 0$ or $\gamma_i = 0$ respectively). The Lagrangian writes:

$$\mathcal{L}(\boldsymbol{b}_{|\mathcal{G}}) = f(\boldsymbol{b}_{|\mathcal{G}}) + \sum_i [\alpha_i(b_{iN} - b_{Ni} - z_i) + \beta_i(b_{iN} - \ell_{iN}) + \gamma_i(b_{Ni} - \ell_{Ni})].$$

The first order conditions with respect to b_{ij} for $(i,j) \in G$, the complementarity conditions and the sign constraints on the multipliers are

$$\begin{aligned}
\text{for each } (i,j) \in \mathcal{G} \quad &: \quad \frac{\partial \mathcal{L}}{\partial b_{ij}} = \log \frac{b_{ij}}{\ell_{ij}} + \alpha_i + \beta_i - \alpha_j + \gamma_j = 0 && (37)\\
\text{for each } i \quad &: \quad \alpha_i \geq 0 \text{ and } \alpha_i W_i = 0 && (38)\\
\text{for each } i \in D \quad &: \quad \beta_i \geq 0 \text{ and } \beta_i(b_{iN} - \ell_{iN}) = 0 && (39)\\
\text{for each } i \notin D \quad &: \quad \gamma_i \geq 0 \text{ and } \gamma_i(b_{Ni} - \ell_{Ni}) = 0 && (40)
\end{aligned}$$

These conditions are necessary and sufficient for an allocation satisfying the constraints of the program to solve $\mathcal{P}_1$. Taking exponential, (37) is equivalent to

$$\begin{aligned}
\text{For each } (i,j) \in \mathcal{G}: \quad & b_{ij} = \delta_i \mu_j \ell_{ij} && (41)\\
\text{where for each } i: \quad & \delta_i = \exp -(\alpha_i + \beta_i) \text{ and } \mu_i = \exp(\alpha_i - \gamma_i). && (42)
\end{aligned}$$

Defining $b_{ij} = 0$ for $(i,j) \notin \mathcal{G}$, i.e. when $\ell_{ij} = 0$, the relation (41) is valid for any (i,j). This proves that $\boldsymbol{b}$ is bi-proportional to $\boldsymbol{\ell}$. It remains to check that a multiplier vector $\boldsymbol{\mu}$ defined by (42) satisfies the rescue conditions (16). Lemma 2 asserts that one can choose $\boldsymbol{\mu}$ to have each of its component above 1 (This may not be true for all multiplier vectors when they are not unique.). We prove that such a $\boldsymbol{\mu}$ satisfies the following properties.

(a) For $i \in D$, $W_i = 0$ and $\mu_i > 1$.

As we have already seen, the non-negativity of W_i for i distressed at a payment-bounded solution implies $b_{iN} - \ell_{iN} < 0$. Because $\mu_j \geq 1$ for each j, $b_{iN} = \delta_i \sum_j \mu_j \ell_{ij} \geq \delta_i \ell_{iN}$ hence we must have $\delta_i < 1$. Furthermore, $\beta_i = 0$ by the complementarity condition (39) so that $\delta_i = \exp -\alpha_i$. We thus obtain $\alpha_i > 0$: $W_i = 0$ by the complementarity condition (38).

(b) $\mu_i = 1$ for each i with $W_i > 0$: (16) is satisfied

Let i with $W_i > 0$ (such i exists). i is surely not distressed from (a) and has $\alpha_i = 0$ by (39). Thus $\mu_i = \exp(-\gamma_i)$ by (42) where $\gamma_i \geq 0$ by (40). Now $\gamma_i > 0$ would imply $\mu_i < 1$: this proves $\gamma_i = 0$ and $\mu_i = 1$.

Finally, creditors' priority is satisfied since, by (a), $W_i = 0$ for $i \in D$ and each $i \notin D$ reimburses fully. This proves that the solution to $\mathcal{P}_1$ is a CbiP-solution.

Conversely, let $\boldsymbol{b}$ be a CbiP-solution. It is feasible for $\mathcal{P}_1$. Since it is written as (41), we show that the first order conditions (37) are satisfied by defining multipliers satisfying (42) as follows:

For $i \notin D$, set $\alpha_i = \ln \mu_i$, $\beta_i = -\ln(\delta_i \mu_i)$ and $\gamma_i = 0$. (42) are satisfied: $\mu_i = \exp(\alpha_i - \gamma_i)$ since $\gamma_i = 0$ and $\delta_i = (\exp - \beta_i)/\mu_i = \exp -(\alpha_i + \beta_i)$.

For $i \in D$ set $\alpha_i = -\ln(\delta_i)$, $\beta_i = 0$ and $\gamma_i = -\ln(\delta_i \mu_i)$. (42) are satisfied: $\delta_i = \exp -\alpha_i = \exp -(\alpha_i + \beta_i)$ since $\beta_i = 0$ and $\mu_i = (\exp(-\gamma_i)/\delta_i = \exp(\alpha_i - \gamma_i)$.

Consider now the complementarity and signs conditions.

For $i \notin D$, $\mu_i \geq 1$ implies $\alpha_i \geq 0$. Furthermore, for i with $W_i > 0$, $\mu_i = 1$ so that $\alpha_i = 0$: the complementarity condition is satisfied. There is no sign condition on β_i and $\gamma_i = 0$ implies the complementarity condition on i's net worth (38).

For $i \in D$, we have seen that necessarily each δ_i is not greater than 1, hence $\alpha_i \geq 0$. $\beta_i = 0$ implies the complementarity condition on i's reimbursement and there is no sign condition on γ_i.

This proves that a CbiP-solution is a solution to $\mathcal{P}_1$, hence is unique. ■

Lemma 2 *The multiplier $\boldsymbol{\mu}$ can be chosen to have $m = \min \mu_i$ at least equal to 1.*

Proof of Lemma 2. Consider a multiplier $\boldsymbol{\mu}$ with m strictly less than 1. Denote $I = \{i, \mu_i = m\}$. The proof follows from several claims.

Claim 1. $b_{Ni} = \ell_{Ni}$ for $i \in I$.

Proof: Let $i \in I$. If $i \in D$, $b_{Ni} = \ell_{Ni}$ by construction. If $i \notin D$, $\mu_i = m < 1$ holds if $\alpha_i < \gamma_i$, hence $\gamma_i > 0$. The complementarity condition (40) then implies $b_{Ni} = \ell_{Ni}$.

Claim 2. Let $i \in I$. For each debtor j of i: $m\delta_j = 1$ and for each creditor k of j: $\delta_j \mu_k = 1$.

Proof: Let $i \in I$. Let us first show that $m\delta_j \leq 1$ for each debtor j of i. Assume by contradiction $m\delta_j > 1$. Thus $\delta_j > 1$, which implies $\alpha_j + \beta_j < 0$ by the definition (42) of δ_j. Hence surely $\beta_j < 0$: j does not belong to D, because of the sign condition

(39). Thus $b_{jN} = \ell_{jN}$, which writes $\sum_k \delta_j \mu_k \ell_{jk} = \sum_k \ell_{jk}$. Since $\ell_{jN} > 0$ (because j is indebted to i), surely $\delta_j \mu_k \leq 1$ for a creditor k of j. Since we have assumed $m\delta_j > 1$, $\delta_j \mu_k \leq 1$ implies $m > \mu_k$: a contradiction.

We derive $m\delta_j = 1$ for each debtor j of i: Since $b_{ij} = \delta_j \mu_j \ell_{ij}$, it follows that $b_{ij} \leq \ell_{ij}$ for each j so that the identity $b_{Ni} = \ell_{Ni}$ (claim 1) can hold only if $m\delta_j = 1$ for each debtor j of i, the first part of the claim.

We thus have that for each debtor j of i: $\delta_j \mu_k = \mu_k / m$ for any k, hence $\delta_j \mu_k \geq 1$ by the definition of m. Since $b_{jk} = \delta_j \mu_k \ell_{jk}$, it follows that $b_{jk} \geq \ell_{jk}$ for each k. Hence $b_{jN} \geq \ell_{jN}$ can hold only if $b_{jk} = \ell_{jk}$ for each k, which implies $.\mu_k / m = 1$ for any k creditor of j: this ends the proof.

End of the proof: Denote by J the set of debtors of I, and by K the set of creditors of J. From Claim 2, $m\delta_j = 1$ for j in J and $\mu_k = m$ for each k in K, hence K is a subset of I. The firms in I have lent only to elements in J who have only borrowed from elements in I. Furthermore the products $\delta_j \mu_i$ are all equal to 1. It follows that we can change each μ_i for $i \in I$ and each δ_j for j in J to 1 without affecting the allocation. Let $\boldsymbol{\mu}'$ be the obtained multiplier vector. The minimum m' of its components is strictly larger than m and furthermore the number of components strictly less than 1 is strictly lower than for μ. If m' is equal to 1, we are done. Otherwise m' is lower than 1, and we can repeat the argument. Since the number of elements with a multiplier strictly less than 1 decreases at each step, multipliers all at least equal to 1 are reached in a finite number of steps. ∎

Proof of Proposition 6. Let $\boldsymbol{b}$ a super-tight solution that satisfies cp-consistency and proportionality in R-ratios. We show that $\boldsymbol{b}$ is bi-proportional to $\boldsymbol{\ell}$ with scales satisfying conditions (16).

Each total reimbursement is fixed: For a non-distressed firm, $b_{iN} = \ell_{iN}$ by contagion-freeness, and for a distressed one,

Step 1. There are $\delta_i \leq 1$ and $\mu^i_j \geq 1$ such that for each $(i,j) \in \mathcal{G} : b_{ij} = \delta_i \mu^i_j \ell_{ij}$ with $\mu^i_j = 1$ if $W_j > 0$. Proof: By cp-consistency, for each i there is $\delta_i \leq 1$ such that $b_{ij} = \max(\delta_i \ell_{ij}, m^i_j)$ where $m^i_j = \max(b_{jN} - z_i - b_{N-i,j}, 0)$. By definition of m^i_j, $W_j = 0$ iff $b_{ij} = m^i_j$, in which case $\delta_i \ell_{ij} \leq m^i_j$. We can thus write $b_{ij} = \delta_i \mu^i_j \ell_{ij}$ where $\mu^i_j \geq 1$ and $\mu^i_j = 1$ if $W_j > 0$: this proves Step 1.

Step 2. μ^i_j is independent of each debtor i of j.

Proof: Consider j with $W_j > 0$: the claim holds since $\mu^i_j = 1$ for each $(i,j) \in \mathcal{G}$. In particular, by assumption, there is a common safe creditor, say 1, for which $W_1 > 0$:

$\mu_1 = 1$. Consider j with $W_j = 0$. R-ratio Let i and k debtors of j. Proportionality in R-ratios applied to these pairs and to $(i, 1)$ and $(k, 1)$ (which are both in $\mathcal{G}$) implies:

$$\frac{b_{ij}}{\ell_{ij}} / \frac{b_{kj}}{\ell_{kj}} = \frac{b_{i1}}{\ell_{i1}} / \frac{b_{k1}}{\ell_{k1}}$$

Plugging the expression $b_{ij} = \delta_i \mu_j^i \ell_{ij}$ for each pair in $\mathcal{G}$ this equation writes

$$\mu_j^i / \mu_j^j = \mu_1^i / \mu_1^k$$

Assuming a common safe creditor, say 1, for which $\mu_1^i = 1$ whatever i, this proves $\mu_j^i = \mu_j^k$ for each (i, j) and (k, j) in $\mathcal{G}$.

End of the proof. Define μ_j to be equal to the common value of the μ_j^i for each (i, j) in $\mathcal{G}$. μ_j is at least equal to 1 since the μ_j^i are. We obtain $b_{ij} = \delta_i \mu_j \ell_{ij}$ for each (j, j) in $\mathcal{G}$ with scales satisfying the condition (16). For (i, j) not in $\mathcal{G}$, $\ell_{ij} = 0$; $b_{ij} = 0$ since $\boldsymbol{b}$ is liability-compatible. Thus $b_{ij} = \delta_i \mu_j \ell_{ij}$ holds for any pair. ∎

Proof of Proposition 7. From Property 2 a tight and bilaterally-bounded solution is of the form $\boldsymbol{b} = (\boldsymbol{b}_{|\mathcal{G}_D}, \boldsymbol{\ell}_{|N^2-\mathcal{G}_D})$, with fixed contributions of the transfers outside $\mathcal{G}_D$ to the entropy measure f. The program $\mathcal{P}_2$ is thus equivalent to

$\mathcal{P}_2$: minimize $f(\boldsymbol{b}_{|\mathcal{G}_D}) = \sum_{(i,j)\in\mathcal{G}_D} b_{ij}[log(\frac{b_{ij}}{\ell_{ij}}) - 1]$ over $\boldsymbol{b}_{|\mathcal{G}_D}$ s.t. $\boldsymbol{b} = (\boldsymbol{b}_{|\mathcal{G}_D}, \boldsymbol{\ell}_{|N^2-\mathcal{G}_D})$ satisfy

$$\text{for each } i: \quad \textstyle\sum_j b_{ij} - \sum_j b_{ji} \leq z_i \tag{43}$$

$$\text{for each } (i,j) \in \mathcal{G}_D: \quad b_{ij} \leq \ell_{ij} \tag{44}$$

A problem in $\mathcal{T}_b^*$ has all conditions (13) satisfied strictly, so the feasible set of $\mathcal{P}_2$ has a non-empty relative interior. The proof follows similar lines as that of Proposition 5 by using the Lagrangean to show that the solutions to convex program $\mathcal{P}_2$ coincide with the CbbiP-solutions, proving both the existence and uniqueness of a CbbiP-solution.

Denote by α_i the Kuhn-Tucker multiplier to i's net worth constraint (43), by β_{ij} the one on i's reimbursement to j (44) for $(i, j) \in \mathcal{G}_D$. The Lagrangian writes:

$$\mathcal{L}(\boldsymbol{b}_{|\mathcal{G}_D}) = f(\boldsymbol{b}_{|\mathcal{G}_D}) + \sum_i [\alpha_i(b_{iN} - b_{Ni} - z_i) + \sum_{(i,j)\in\mathcal{G}_D} \beta_{ij}(b_{ij} - \ell_{ij}).$$

The first order conditions and the complementarity conditions are

$$\begin{aligned}
\text{For each } (i,j) \in \mathcal{G}_D: \quad & \frac{\partial \mathcal{L}}{\partial b_{ij}} = \log\frac{b_{ij}}{\ell_{ij}} + \alpha_i + \beta_{ij} - \alpha_j \leq 0 \text{ with } = \text{ if } b_{ij} < \ell_{ij} && (45)\\
\text{for each } i: \quad & \alpha_i \geq 0 \text{ and } \alpha_i(z_i + b_{Ni} - b_{iN}) = 0 && (46)\\
\text{for each } i,j \in \mathcal{G}_D: \quad & \beta_{ij} \geq 0 \text{ and } \beta_{ij}(b_{ij} - \ell_{ij}) = 0 && (47)
\end{aligned}$$

Let us first show that the solution to $\mathcal{P}_2$ is super-tight. It suffices to show that it satisfies minimal rescue. By contradiction, assume $W_i > 0$ for $i \in D$. Then $\alpha_i = 0$ by the complementarity condition (46). We prove that $b_{ij} = \ell_{ij}$ for each j. For j in D, this holds by construction. For j not in D, the pair (i,j) is in $\mathcal{G}_D$. $b_{ij} < \ell_{ij}$ would imply $\beta_{ij} = 0$ (from 47) hence 45) writes $\log\frac{b_{ij}}{\ell_{ij}} = \alpha_j$, which contradicts $b_{ij} < \ell_{ij}$ since $\alpha_j \geq 0$. This proves that distressed i reimburses fully its creditors: its net worth must be strictly negative, a contradiction.

To show that $\boldsymbol{b}$ is the CbbiP-solution, from Lemma 3, it suffices to define δ_i for i in D and μ_j for j not in D such that (18) is satisfied on $\mathcal{G}_D$ and the rescue conditions (16) are satisfied on D^c.. Set for i in D: $\delta_i = \exp -\alpha_i$ and for j in D^c: $\mu_j = \exp \alpha_j$.

$\boldsymbol{b}_{|\mathcal{G}_D}$ *satisfies (18).* From (45) we obtain

$$\text{for each } i,j \in \mathcal{G}_D : b_{ij} = \exp -(\beta_{ij})\delta_i\mu_j\ell_{ij}. \tag{48}$$

There are two cases. Case 1: $\beta_{ij} > 0$. In that case, $b_{ij} = \ell_{ij}$ by the complementarity condition (47) and $\exp -(\beta_{ij}) < 1$. Thus (48) implies $\ell_{ij} < \delta_i\mu_j\ell_{ij}$, hence $\ell_{ij} = \min(\delta_i\mu_j, 1)\ell_{ij}$, which proves (18). Case 2: $\beta_{ij} = 0$. In that case, $b_{ij} = \delta_i\mu_j\ell_{ij}$. Surely $\delta_i\mu_j \leq 1$ since $b_{ij} \leq \ell_{ij}$ by (44). Hence $b_{ij} = \min(\delta_i\mu_j, 1)\ell_{ij}$: again (18) is satisfied.

μ_j *satisfies (16) for j in D^c.* Since $\mu_j = \exp \alpha_j$ and $\alpha_j \geq 0$ obviously each μ_j is larger than 1 and $\mu_j > 1$ only if $\alpha_j > 0$ in which case $W_j = 0$ by the complementarity condition (46): this proves (16).

The solution to $\mathcal{P}_2$ thus satisfies all the constraints required on a CbbiP-solution. This proves the existence of a CbbiP-solution. Conversely, a CbbiP-solution solves $\mathcal{P}_2$: it belongs to the feasible set of $\mathcal{P}_2$ and it is easy to define the multipliers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ from $\boldsymbol{\delta}$ and $\boldsymbol{\mu}$ so that the first order conditions and complementarity conditions of the Lagrangian are met. This proves the uniqueness of a CbbiP-solution. ■

Lemma 3 *Let $\boldsymbol{b}$ be bilaterally-bounded and super-tight. Let δ_i for i in D and μ_j for j not in D such that (18) is satisfied on $\mathcal{G}_D$ and the rescue condition (16) is satisfied* *on D^c. Then defining $\delta_i = 1$ for i not in D and $\mu_j = \max_{k\in D} 1/\delta_k$, (18) is satisfied for any pair and the rescue condition (16) for any j.*

Proof The definition of μ_j for $j \in D$ implies $\mu_j$is larger than 1 since surely $\delta_i \leq 1$ for i in D. This is compatible with (16) since $\boldsymbol{b}$ satisfies minimal rescue hence $W_j = 0$. Consider now (18) for a pair (i,j) not in $\mathcal{G}_D$. Either i non-distressed (hence $\delta_i = 1$) or both i and j distressed, hence $\mu_j \geq 1/\delta_i$. It straightforwardly follows that for each (i,j) not in $\mathcal{G}_D$: $\delta_i\mu_j \geq 1$, hence $\ell_{ij} = \min(\delta_i\mu_j, 1)\ell_{ij}$. Since $b_{ij} = \ell_{ij}$ this proves (18). ∎

Proof of Proposition 8. Let $\boldsymbol{b}$ be bilaterally-bounded and super-tight and satisfy cp-consistency and constrained proportional R-ratios. We show that $\boldsymbol{b}$ is the (unique) CbbiP-solution, following the same lines as in the proof of Proposition 6. In particular, Since cp-consistency holds on $\mathcal{G}_D$, step 1 is straightforwardly changed into:

Step 1. There are δ_i for $i \in D$ and $\mu_j^i \geq 1$ for $j \notin D$ such that for each $(i,j) \in \mathcal{G}_D$: $b_{ij} = \delta_i\mu_j^i\ell_{ij}$ with $\mu_j^i = 1$ if $W_j > 0$. Surely $\delta_i < 1$.

Step 2. (18) is satisfied on $\mathcal{G}_D$: for each $(i,j) \in \mathcal{G}_D : b_{ij} = \min(\delta_i\mu_j, 1)\ell_{ij}$.

Proof. Let j in D^c with $W_j > 0$. From Step 1, $\mu_j^i = 1$ for each $(i,j) \in \mathcal{G}_D$ hence $b_{ij} = \delta_i\ell_{ij}$. Define $\mu_j = 1$. Since $\delta_i \leq 1$, we have $\min(\delta_i\mu_j, 1) = \delta_i$: (18) is satisfied. In particular it is satisfied for a common safe creditor, say 1.

Let j in D^c with $W_j = 0$. Constrained Proportionality in R-ratios implies that for each pair (i,j) and (k,j) both in $\mathcal{G}_D$: $\frac{b_{ij}}{\ell_{ij}}/\frac{b_{kj}}{\ell_{kj}} < \frac{b_{i1}}{\ell_{i1}}/\frac{b_{kl}}{\ell_{kl}}$ implies $b_{ij} = \ell_{ij}$. Plugging the expression $b_{ij} = \delta_i\mu_j^i\ell_{ij}$ and using $\mu_1^i = \mu_1^k = 1$, the condition writes:

$$\mu_j^i < \mu_j^k \text{ implies } b_{ij} = \ell_{ij}$$

Define $I = \{k \text{ s.t. } b_{kj} < \ell_{kj}\}$. The above condition implies that the values of μ_j^k are equal on I and furthermore not less than any μ_j^i. Setting μ_j equal to the maximum of the μ_j^i over all i we have $b_{kj} = \delta_k\mu_j\ell_{kj}$ with $\delta_k\mu_j < 1$ for each $k \in I$: (18) is satisfied. For i not in I, $b_{ij} = \ell_{ij}$, hence $\delta_i\mu_j^i = 1$. Since $\mu_j^i \leq \mu_j$, we obtain $\delta_i\mu_j \geq 1$, which gives $b_{ij} = \min(\delta_i\mu_j, 1)\ell_{ij}$. This proves that conditions (18) are satisfied on $\mathcal{G}_D$. It is obvious that (16) is satisfied. Using Lemma 3, this ends the proof. ∎