STRATEGYPROOFNESS IN THE ASSIGNMENT MARKET GAME

by Gabrielle DEMANGE

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ÉCOLE POLYTECHNIQUE

LABORATOIRE D'ÉCONOMÉTRIE

5, rue Descartes - 75230 Paris cedex 05 - FRANCE

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I INTRODUCTION

Shapley and Shubik studied in [4] a market where one good was present in indivisible units and they proved the interesting fact that core and set of competitive equilibria coincide in this market and are never empty. The main features of the market are that 1) there are two kinds of agents - buyers and sellers -, 2) each seller owns one unit of the indivisible commodity and each buyer has no use for no more than one unit, 3) utility is transferable and side payments are allowed. For example, a market of houses may meet these assumptions. Competitive equilibria provide here efficient, stable, envy-free - no buyer envies the situation of another one - allocations, but the problem of reaching such equilibria has not yet be solved: the result of a market process with contacts, bids, offers is by no means clear since there is an infinity of equilibria (1) and last but not least tactical interests may incite agents to misrepresent their true utility valuation.

One aim of the paper is to present a mechanism which implements one specific equilibrium in a very convenient way: truth telling is a dominant strategy for each buyer and an optimal one (in the maximin sense) for each seller. This means that when sellers have no information on buyers and are prudent the outcome of the mechanism is a competitive equilibrium with

(1) There exists generically a unique optimal assignment of the items but an infinity of equilibrium prices.

respect to the <u>true</u> utility valuations. The result relies on the special structure of the set of competitive equilibria; one of the equilibria is the best one for <u>each</u> buyer since it corresponds to a minimal price vector: the price of each object is the minimal it can be in an equilibrium. The mechanism which we propose implements this very equilibrium which most favors every buyer. The same property is valid in the well-known assignment game where money is excluded: in that case - often called the marriage problem - there is still a stable assignment which most favors every individual of one side and the Gale-Shapley algorithm allows to reach this assignment without manipulation of the most favoured side (see Dubins and Freedman [1]).

The second purpose of the paper is to present an auction which is strategically equivalent to the presented mechanism but is of computational interest since it allows to find an optimal assignment and the minimal price vector at the same time.

To illustrate the results, I present now them in the simple case where there is just one seller, that is one object to sell. The mechanism works as follows:

- first, the seller announces a positive number, say d;
- second, each buyer announces a number; for example, if there are n buyers ranged by decreasing bids: $b_1 \geqslant b_2 \geqslant \dots \geqslant b_n$.

The outcome is:

if $b_1 \geq 0$ buyer one gets the object and pays to the seller $b_2 + d$ if $b_2 \geq 0$ or d if $b_2 \leq 0$; other buyers pay nothing.

if $b_1 < 0$ there is no transaction.

This means that d is interpreted as the minimum price at which the seller accepts to sell his object and $b_j + d$ as the maximum price at which buyer j accepts to buy the object; thus truth telling is understood as announcing for the seller exactly how much he values his object and for a buyer his true surplus of valuation with respect to the valuation announced by the seller. But once d has been announced, buyers play exactly the Vickrey's game [5]: the object is given to the highest bidder at the second highest bid; it is well-known that to tell the truth is a dominant strategy in this game. Moreover truth telling obviously ensures to the seller a zero profit and no other strategy ensures him a higher one. Therefore, if c and $(h_j)_{1 \leq j \leq n}$ are the valuations of the seller and buyers and if the seller announces c, the outcome will be: buyer 1 gets the object and pays for it h_2 (we suppose $h_2 \geq c$) to the seller. One easily verifies that the equilibrium allocations are all the allocations which give the object to 1 (we suppose $h_1 > h_2$) at a price between h_2 and h_1 thus the equilibrium resulting from the game is the best one for the buyers. Thus our first result is a generalization of this one. Moreover it is well-known that the Vickrey's rule is equivalent to the American auction where the price of the object is raised until every buyer but one drops out. The auction we propose appears to be a generalization of the American auction for dealing with the case of several objects.

The paper is organized as follows: definitions, description of the model are given in part II, mechanisms are studied in part III, proof of a lemma in part IV.

II MODEL AND DEFINITIONS

1. The two-sided market

We consider the two-sided market introduced by Shapley and Shubik in [4]:

- there are two kinds of agents, called buyers and sellers; the set of buyers is denoted by $N = \{1, \ldots, j, \ldots, n\}$ and the set of sellers by $M = \{1, \ldots, i, \ldots, m\}$;
- each seller owns one unit of an indivisible commodity; a unit is indexed by the same subscript as its owner; all units are of the same type but not necessarily alike so that each buyer has no use for more than one object but may value each object differently;

- side-payments are allowed and utility is transferable; buyer j (resp. seller i) values object i at h_{ij} (resp. c_i) units of money.

This market will be denoted by (c, H) where c is the m-vector $(c_i)_{i \in M}$ and H the mxn matrix $(h_{ij})_{i \in M,\, j \in N}$; A is the mxn matrix $(a_{ij})_{i \in M,\, j \in N}$ where $a_{ij} = h_{ij} - c_i$; max $(a_{ij}, 0)$ may be interpreted as the value of the coalition $\{i,j\}$.

2. Equilibrium price vector; core

The final result of exchanges of objects between buyers and sellers is represented by an assignment: an assignment is a one to one mapping from a subset of N onto a subset of M; Σ denotes the set of the assignments; if σ is in Σ, N_σ (resp. M_σ) denotes its domain (resp. its range); note that $|N_\sigma| = |M_\sigma|$. Σ_j denotes the set of assignments which give nothing to agent j : $\Sigma_j = \{\sigma \in \Sigma,\ j \notin N_\sigma\}$. An assignment σ generates a total surplus of utility measured by $v(\sigma) = \sum_{j \in N_\sigma} a_{\sigma(j)j}$; we shall use the notation $v_k(\sigma) = \sum_{\substack{j \in N_\sigma \\ j \neq k}} a_{\sigma(j)j}$ where k is in N_σ.

We define v(N) as $\max_{\sigma \in \Sigma} v(\sigma)$ and $v(N-\{k\})$ as $\max_{\sigma \in \Sigma_k} v(\sigma)$. An assignment σ is optimal if $v(\sigma) = v(N)$.

A price vector $(p_i)_{i \in M}$ is a vector in R_+^m where p_i represents the price of object i.

An <u>allocation</u> is a couple (σ, p) where σ is an assignment and p a price vector; it has to be interpreted as follows: a buyer j in N_σ receives object $\sigma(j)$ and pays $p_{\sigma(j)}$ to seller $\sigma(j)$, agents neither in N_σ nor in M_σ pay and receive nothing.

If p is a price vector and h an m-vector D(p,h) represents the <u>demand</u> at prices p of a buyer whose valuation is h: $D(p,h) = \{i \in M \cup \{0\},\ h_i - p_i = \max_{k \in M\cup\{0\}} (h_k - p_k)\}$, where 0 is a fictitious object which represents the null trade; by convention $h_o = p_o = 0$.

An <u>equilibrium allocation</u> (σ,p) of the market (c,H) is an allocation such that:

- for every j in N_σ $\quad \sigma(j)$ belongs to $D(p,h_j)$
- for every j not in N_σ 0 belongs to $D(p,h_j)$
- for every i in M $\quad p_i \geq c_i$ with equality when i is not in M_σ.

This means that at prices p each buyer gets an object - eventually the null trade - which yields him the higher utility level, each seller gets a positive profit and each object not sold is of minimal price. To every allocation (σ,p) are associated profit vectors (u,v) in R^{m+n} by:

$u_i = p_i - c_i$ which represents the profit of seller i

$v_j = h_{\sigma(j)j} - p_{\sigma(j)}$ if j is in N_σ.

$v_j = 0$ if j is not in N_σ.

One checks that, if (σ,p) is an equilibrium allocation, then σ is an optimal assignment and that for every other optimal assignment σ', if any, (σ',p) is <u>also</u> an equilibrium of the market and it yields to the same profit vector (this point is proved in IV 1.c), thus we may speak of <u>equilibrium price vector</u> (e.p.v) and of the profit vector associated with, without specifying which particular optimal assignment we consider.

Shapley and Shubik proved in [4] that, given the market (c, H),

* p is an equilibrium price vector of the market iff the associated profit vector (u,v) is in the core of the market game;

* there exists an equilibrium vector (which is not obvious since indivisibilities are present) and moreover there exists a <u>minimal</u> equilibrium price vector, say $p_\star$, i.e. for every other e.p.v. p, $p \geqslant p_\star$ (1) or equivalently a minimal profit vector u for the sellers since $u = p - c$.

<u>Remark</u>. Since $D(p,h) = D(p-c, h-c)$ if $p \geqslant c$, a vector p is an e.p.v of the market (c,H) if the vector $u = p - c$ is an e.p.v of the market (0,A); thus, when c is known, we may indifferently work with H and p or with A and u. In the sequel we shall always work with A and u and speak (incorrectly) of minimal equilibrium price vector $u_\star$.

(1) We use the following notation for vector inequalities in R^m:
$x \geqslant y$ means $x_i \geqslant y_i$ $i = 1, \ldots, m$
$x \gg y$ means $x_i > y_i$ $i = 1, \ldots, m$

II THE MECHANISM

1. The Vickrey mechanism

The first mechanism is simple to describe and we call it the Vickrey mechanism since it is a generalization of the Vickrey's rule to the multiple objects case. The game consists of two steps:

first step: sellers announce simultaneously a positive number, say d_i for seller i;

second step: buyers announce simultaneously a m-vector, say b_j for buyer j;

outcome: if σ and $u_\star$ are respectively an optimal assignment and the minimal equilibrium price vector of the market (0, B) where B is the mxn matrix whose column j is b_j, buyer j in N_σ receives object $\sigma(j)$ and pays $d_{\sigma(j)} + u_{\star\sigma(j)}$ to seller $\sigma(j)$, buyers not in N_σ receive and pay nothing, sellers not in M_σ keep their object.

Remark: In order to well define the game, we ought to first choose a selection of the optimal assignment (recall that however optimal assignment is generically unique), but we shall prove that truth is announced at equilibrium and so all optimal assignments are equivalent in that case.

The interpretation of d and B is the same as in the one seller's case given in the introduction, so in the market (c, H) truth telling for seller i consists of announcing $d_i = c_i$ and for buyer j, $b_j = h_j - d$. Thus, if all the agents announce the truth, the outcome is an equilibrium allocation and more precisely the best equilibrium allocation for the buyers. This explains the interest of the following result:

Theorem 1

In the Vickrey mechanism, truth telling is an optimal strategy in the maximin sense for every seller and a dominant strategy for every buyer.

The theorem is proved in section 3.

2. The auction mechanism

The auction relies on the Hungarian algorithm suitably specified; this algorithm is used to find an optimal assignment relative to a given matrix with the help of dual variables interpreted as prices. Suppose the matrix to be an mxn matrix $A = (a_{ij})_{i \in M,\ j \in N}$. The dual variable is a m-vector, denoted by p(t) at the end of step t; initial price vector p(0) is zero. The algorithm begins at step (1) (t=0) and step (t+1), if it exists, is defined as follows:

step (t+1): if there exists an assignment σ in Σ such that $\sigma(j)$ belongs to D $(p(t), a_j)$ for every j in N_σ and 0 belongs to D $(p(t), a_j)$ for every j not in M_σ, the algorithm stops.

If such an assignment is not possible, there exists some "overdemanded" set, i.e. a set S, $\emptyset \neq S \subset M$, such that for some subset T of N:

. $D(p(t), a_j) \subset S$ for every j in T

. $|T| > |S|$

Choose such a couple S, T with S <u>minimal</u> (with respect to the cardinality) and raise the prices of the objects of S by the same smallest ε such that, for at least one j in T, the demand associated to a_j changes, i.e. contains either the null trade or an object not in S since the relative prices of objects in S do not change. This defines p(t+1) and go to step (t+2).

The only difference with the usual Hungarian algorithm is that we require the minimality of the chosen overdemanded set; so we already know that the algorithm is convergent and stops at a price p which is an equilibrium price vector relative to A (see for example Lawler [3]); by requiring the minimality condition the final price is in fact the minimal equilibrium price vector relative to A (see lemma below). This suggests to replace the second part of the Vickrey mechanism by an auction: an auctioneer asks for the demand sets at prices 0; if possible he assigns the objects such that every agent gets an object demanded and the final allocation is such an assignment with price vector equal to d; if not, the auctioneer chooses an overdemanded set S, $S \subset M$ and a corresponding set T (one may suppose an order on the couples (S, T), $S \subset M$, $T \subset M$ to be given), raises continuously the prices of the objects in S until some individual in T stops the auctioneer, demand sets are asked at this new price vector and so on. To define the game we must define the set of admissible strategies: a strategy for a player is a function which assigns to each price vector a non-empty subset of $M \cup \{0\}$ which is interpreted as the demand set of the player at those prices; this suggests that some restrictions are to be required: for example if at prices (0) an individual demands object 1 he must still demand it as long as the price of object 1 is zero, or the demands must depend only on the relative prices as long as the null trade is not demanded; one can show that this entails the existence of a m-vector b such that the strategy is the demand associated to the valuation vector b; one may argue that these restrictions are too demanding since in fact the auctioneer observes the demands only at some price vectors and not at all price vectors; this leads to require the consistency conditions only for the prices which appear in the auction but then the admissibility of an individual strategy depends on the strategies of other players; since we suppose players to be non-cooperative we exclude such possibility.

Theorem 2

The Vickrey mechanism and the auction are equivalent.

3. Lemma, proofs and example

The theorems 1, 2 rely on the following lemma proved in part IV :

3.a. Lemma

Let A be a mxn matrix. The following statements are equivalent:

(1) $u_\star$ is the minimal equilibrium price vector relative to A.

(2) if σ is an optimal assignment relative to A

$u_{\star i} = 0$ if i is not in M_σ

$u_{\star\sigma(j)} = v\ (N-\{j\}) - v_j(\sigma)$ for every j in M_σ

(3) the Hungarian algorithm suitably defined (see III.2) converges to $u_\star$.

3.b. Proof of theorem 1

a) Truth telling is an optimal strategy for a seller: by announcing $d_i = c_i$ seller i ensures a zero profit since either he does not sell (zero profit) or he sells at $c_i + u_{\star i}$ and gets a profit of $u_{\star i} \geq 0$. Obviously no strategy d_i ensures him a strictly positive profit since, when $h_{ij} < d_i$ for each j the object is not sold.

b) Truth telling is a dominant strategy for each buyer; this follows directly from the assertion (2) of the lemma: consider w.l.o.g. buyer 1 and let a_j the vector announced by buyer j, j = 2, ..., n; we denote by σ, $u_\star$ the outcome if buyer 1 announces the truth, i.e. $a_1 = h_1 - d$ and by σ', $u'_\star$ the outcome if he announces another vector, say b_1. In first case his final utility level v_1 is:

$h_{\sigma(1)1} - d_{\sigma(1)} - u_{\star\sigma(1)} = a_{\sigma(1)1} - u_{\star\sigma(1)}$ if 1 is in N_σ and 0 if 1 is not in N_σ

In second case, his final utility level v_1' is:

$h_{\sigma'(1)1} - d_{\sigma'(1)} - u'_{\star\sigma'(1)} = a_{\sigma'(1)1} - u'_{\star\sigma'(1)}$ if 1 is in $N_{\sigma'}$ and 0 if 1 is not in $N_{\sigma'}$

If 1 belongs to N_σ, v_1 is equal (by lemma) to:

$$a_{\sigma(1)1} - v\ (N-\{1\}) + v_I(\sigma) = v(N) - v\ (N-\{1\})$$

where $v(N) = \sum_{j \in N_\sigma} a_{\sigma(j)j} = \max \sum_{j \in N_\tau} a_{\tau(j)j}$

and $v\ (N-\{1\}) = \max_{\tau \in \Sigma_I} \sum_{j \in N_\tau} a_{\tau(j)j}$

If 1 belongs to $N_{\sigma'}$, v_1' is equal to:

$$\sum_{j \in N_{\sigma'}} a_{\sigma'(j)j} - v\ (N-\{1\}) = v(\sigma') - v\ (N-\{1\})$$

($v\ (N-\{1\})$ does not change since it depends only on a_j, $j \geqslant 2$)

We consider now four cases to evaluate $v_1 - v_1'$:

. If 1 belongs to N_σ and $N_{\sigma'}$:

$v_1 - v_1' = v(N) - v(\sigma')$ which is positive by definition of $v(N)$

. If 1 belongs to N_σ but not to $N_{\sigma'}$, $v_1 - v_1' = v_1$ and v_1 is always positive.

. If 1 belongs to $N_{\sigma'}$ but not to N_σ, $v_1 - v_1' = -v_1'$ $= -v(\sigma') + v\ (N-\{1\})$; 1 does not belong to N_σ implies $\sigma \in \Sigma_I$ and $v(\sigma) \leqslant v\ (N-\{1\})$, since $v(\sigma) = v(N)$ and $v\ (N-\{1\})$ is always smaller than $v(N)$ we get $v(N) = v\ (N-\{1\})$; therefore $v_1 - v_1'$ $= -v(\sigma') + v(N) \geqslant 0$.

. If 1 belongs neither to N_σ nor $N_{\sigma'}$, $v_1 - v_1' = 0$. Thus in every case $v_1 - v_1' \geq 0$ and truth is a dominant strategy.

Remarks: 1) Once the sellers have announced d, the outcome may be viewed as an optimal collective decision (the assignment) among buyers plus some transfers (the prices); the assertion (2) of the lemma says that these transfers are of the Vickrey Groves' type and utility being transferable it is well-known that the procedure is then strategy-proof. Thus what is new in the theorem is that those transfers yield to a competitive equilibrium.

2) By using the caracterization of the strategy-proof mechanisms (Green-Laffont [2]), one can show that the above mechanism played by the buyers is the only one which is strategy-proof optimal and individually rational.

3.c. Proof of theorem 2

Once d has been announced, buyers choose strategies of the form $D(p,a)$, a in R^m and the outcome of the game when the n-tuple of strategies is $(D(p,a_j))_{j\in N}$ is the same by assertion (3) of the lemma as the outcome of the Vickrey mechanism when $(a_j)_{j\in N}$ is announced. The games are thus perfectly equivalent.

3.d. Example

Suppose the matrix A to be:

$M = N = \{1, 2, 3\}$

8	6	7
7	8	6
5	6	5

There is a unique optimal assignment σ: $\sigma(j) = j$, $j = 1, 2, 3$ and $v(\sigma) = 8 + 8 + 5 = 21$; to compute $u_\star$ with the formula (2) of lemma, one must find the values of coalitions of two players:

$$v(\{1,2\}) = 8 + 8 = 16$$

$$v(\{1,3\}) = 7 + 7 = 14$$

$$v(\{2,3\}) = 8 + 7 = 15$$

Thus $u_{\star\sigma(1)} = u_{\sigma 1} = v(\{2,3\}) - v_1(\sigma) = 15 - 13 = 2$

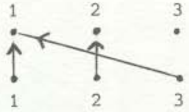
$$u_{\star\sigma(2)} = u_{\sigma 2} = v(\{1,3\}) - v_2(\sigma) = 14 - 13 = 1$$

$$u_{\star\sigma(3)} = u_{\sigma 3} = v(\{1,2\}) - v_3(\sigma) = 16 - 16 = 0$$

To easily describe the successive steps of the algorithm, we represent at each step the demand graph: an arrow is drawn from a buyer to an object if the buyer demands it.

Step 1: demand graph at $p(0)$: objects

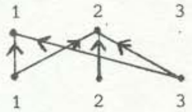
$p(0) = (0, 0, 0)$ buyers



There is a unique minimal overdemanded set: $S = \{1\}$; $T = \{1,3\}$; price of object 1 is raised until 1.

Step 2: demand graph at $p(1)$: objects

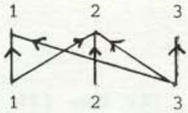
$p(1) = (1, 0, 0)$ buyers



There is a unique oberdemanded set: $S = \{1,2\}$; $T = \{1,2,3\}$; price of objects 1, 2 are raised until 1.

Step 3: demand graph at p(2): objects

$p(2) = (2, 1, 0)$ buyers



There is a possible assignment σ: $\sigma(j) = j$ $j = 1, 2, 3.$

The algorithm stops and $p(2) = u_\star$.

We consider the same valuation matrix as in p. 15.

There is a unique optimal assignment σ :
$\sigma(j) = j$, $j = 1, 2, 3$. $u_\star = (2, 1, 0)$, the associate graph is :

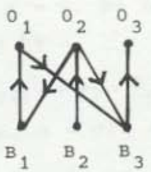


Figure 1

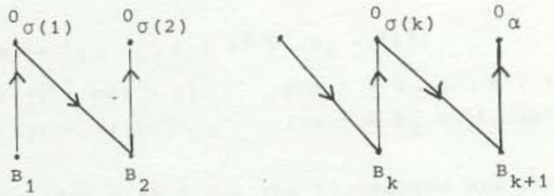


Figure 2

not in T. Moreover an agent j in T still demands $\sigma(j)$ at u since the price of $O_{\sigma(j)}$ has been lowered ; thus u is an equilibrium price vector which is in contradiction with the minimality of $u_{\star}$.

So we may consider a path from $0_{\sigma(1)}$ to 0_{α} where $u_{\star\alpha} = 0$ or $\alpha = 0$. We may suppose the successive nodes of the path to be :

$$0_{\sigma(1)}, B_2, 0_{\sigma(2)}, \dots, B_{k+1}, 0_{\alpha}$$

where either $\alpha = \sigma(k+1)$, $u_{\star\alpha} = 0$ and every buyer B_j, $j = 2, \dots, k+1$, is in N_{σ} or $\alpha = 0$ and only buyers $j = 2, \dots, k$ are in N_{σ}.

Remark that B_1 is not in the path (see figure 2).

Consider now the assignment τ in $\Sigma_{\dagger}$ defined as follows :

$$N_{\tau} = \{2, \dots, k+1\} \cup (N_{\sigma} - \{1\})$$
$$\tau(j) = \sigma(j-1) \qquad \text{for } j = 2, \dots, k+1$$
$$\tau(j) = \sigma(j) \qquad \text{for } j \in N_{\sigma} - \{1, \dots, k+1\}$$

Thus τ coincides with σ for the buyers who are not in the path except for B_1 which receives nothing and τ assigns to each buyer B_j in the path an object of his demand set (since $(0_{\sigma}(j-1), B_j)$ is an edge) but different from $\sigma(j)$. Thus $v(\tau) - v_{\dagger}(\sigma)$ is equal to :

$$\sum_{2 \leqslant j \leqslant k} (a_{\sigma(j-1)j} - a_{\sigma(j)j}) + \begin{cases} a_{\sigma(k)k+1} & \text{if } \alpha = 0, \text{i.e. if } k+1 \notin N_{\sigma} \\ a_{\sigma(k)k+1} - a_{\sigma(k+1)k+1} & \text{if } \alpha \in M \end{cases}$$

But for every $2 \leqslant j \leqslant k$ $a_{\sigma(j-1)j} - a_{\sigma(j)j} = u_{\star\sigma(j-1)} - u_{\star\sigma(j)}$ since B_j demands at price $u_{\star}$ both objects $\sigma(j-1)$ and $\sigma(j)$.

Moreover :

- if $\alpha = 0$ B_{k+1} demands at $u_\star$ the null trade and object $\sigma(k)$ which entails $a_{\sigma(k)k+1} = u_{\star\sigma(k)}$

- if $\alpha \in M$ then $u_{\star\sigma(k+1)} = 0$

and $a_{\sigma(k)k+1} - a_{\sigma(k+1)k+1} = u_{\star\sigma(k)}$.

So in both cases we get :

$$v(\tau) - v_I(\sigma) = \sum_{2\leq j\leq k} u_{\star\sigma(j-1)} - u_{\star\sigma(j)} + u_{\star\sigma(k)}$$

$$= u_{\star\sigma(1)}$$

Since τ is in Σ_I $v(\tau)$ is smaller than $v(N - \{1\})$ and we obtain $v(N - \{1\}) - v_I(\sigma) \geq u_{\star\sigma(1)}$.

■

b. $u_{\star\sigma(j)} \geq v(N - \{j\}) - v_j(\sigma) \quad \forall$ j in N_σ

w.l.o.g. suppose j = 1 and choose τ in Σ_I

$$v(\tau) - v_I(\sigma) = \sum_{j\in N_\tau} a_{\tau(j)j} - \sum_{\substack{j\in N_\sigma \\ j\neq 1}} a_{\sigma(j)j}$$

By using the following inequalities

if j is in $N_\tau \cap N_\sigma$ $a_{\tau(j)j} - a_{\sigma(j)j} \leq u_{\star\tau(j)} - u_{\star\sigma(j)}$

if j is in $N_\tau - N_\sigma$ $a_{\tau(j)j} \leq u_{\star\tau(j)}$

if j is in $N_\sigma - N_\tau$ $a_{\sigma(j)j} \geq u_{\star\sigma(j)}$

we get $v(\tau) - v_I(\sigma) \leq \sum_{j\in N_\tau} u_{\star\tau(j)} - \sum_{\substack{j\in N_\sigma \\ j\neq 1}} u_{\star\sigma(j)}$

Since τ is one to one and each object not in M_σ is of null price, $\sum_{j \in N_\tau} u_{\star\tau(j)} \leq \sum_{j \in N_\sigma} u_{\star\sigma(j)}$

so $v(\tau) - v_I(\sigma) \leq u_{\star\sigma(1)}$

and $v(N - \{1\}) = \max_{\tau \in \Sigma_I} v(\tau)$ implies the inequality

$$v(N - \{1\}) - v_I(\sigma) \leq u_{\star\sigma(1)}$$

■

c. $u_i = 0$ if i is not in M

This relies on the fact stated in II that an equilibrium price vector sustains every optimal assignment and consequently every object which is not assigned in an optimal assignment is of null price. Since we have just stated this property we prove it now : Suppose that σ , σ' are optimal assignments and that the price vector u satisfies :

$0 \leq a_{\sigma(j)j} - u_{\sigma(j)} = \max_{i \in M} (a_{ij} - u_i)$ for each j in N_σ

$a_{ij} \leq u_i$ for each j not in N_σ and each i in M

$u_i = 0$ if $i \notin M_\sigma$

This implies

$$a_{\sigma'(j)j} - a_{\sigma(j)j} \leq u_{\sigma'(j)} - u_{\sigma(j)} \quad \text{for } j \text{ in } N_\sigma \cap N_{\sigma'}$$

$$a_{\sigma'(j)j} \leq u_{\sigma'(j)} \quad \text{for } j \text{ in } N_{\sigma'} - N_\sigma$$

$$- a_{\sigma(j)j} \leq - u_{\sigma(j)} \quad \text{for } j \text{ in } N_\sigma - N_{\sigma'}$$

Thus

$$0 = v(\sigma') - v(\sigma) \leq \sum_{j \in N_{\sigma'}} u_{\sigma'(j)} - \sum_{j \in N_\sigma} u_{\sigma(j)}$$

But $\sum_{j \in N_{\sigma'}} u_{\sigma'(j)} \leq \sum_{j \in N_\sigma} u_{\sigma(j)}$ since every object not in M_σ is of null price.

So all the inequalities are in fact equalities and moreover every object which is not in $M_\sigma \cap M_{\sigma'}$ is of null price. This proves the result.

■

2. Proof of the equivalence of (1) and (3).

We already know the algorithm to be convergent to a vector p with $p \geq u_\star$ since p is an equilibrium price vector. We will prove by induction on the number of steps that $p \leq u_\star$. We denote by p(t) the price vector at the beginning of step t + 1, t = 0 , 1 , ...

- at step 1 , p(0) = 0 and $p(0) \leq u_\star$.

- suppose $p(t) \leq u_\star$; if the price vector is not changed, there is nothing to prove since p = p(t) ; if not, the auctioneer chooses two sets S , T with $S \subset M$, $T \subset N$ and :

+ $|T| > |S|$, $D(p(t), T) = S$

++ there is no subset $S_1 \neq \emptyset$, $S_1 \subset S$ such that $|T_1| \leqslant |S_1|$ where $T_1 = \{j \in T, D(p(t), j) \cap S_1 \neq \emptyset\}$.

The condition ++ is merely implied by the minimality of the overdemanded set S : if there were S_1 with $|T_1| \leqslant |S_1|$ the set $S - S_1$ distinct from S ($S_1 \neq 0$) and from $\emptyset$ ($S_1 = S$ is impossible) would be overdemanded :

$$|T_1| \leqslant |S_1| \quad \text{where} \quad T_1 = \{j \in T, d(p(t), j) \cap S_1 \neq \emptyset\}$$

The prices of the objects in S are raised simultaneously until the demand of an agent in T changes, i.e. contains an element in $M \cup \{0\}$ which is not in S. This means that $p(t + 1) = p(t) + \varepsilon\, e_S$, $\varepsilon > 0$ where e_S is the vector of $\mathbb{R}^m$ whose i-th coordinate is 1 if i is in S and 0 if not, and that for ε' , $0 < \varepsilon' < \varepsilon$ $D(p(t) + \varepsilon' e_S, j) = D(p(t), j)$ for every j in T.

Suppose now that $p(t + 1) \leqslant u_\star$ is not true ; the set S_1 : $S_1 = \{i \in M, u_{\star i} < p_i(t + 1)\}$ is therefore a non-empty subset of S and we can choose ε' , $0 < \varepsilon' < \varepsilon$ such that the vector q : $q = p(t) + \varepsilon' e_S$ satisfies :

- $q|_{S_1} \geqslant u_\star|_{S_1}$, $q|_{S_1^c} \leqslant u_{\star}{}_{S_1^c}$

- $D(q, T_1) = D(p(t), T_1)$ where

$T_1 = \{j \in T, D(p(t), j) \cap S_1 = \emptyset\}$

By property (++), $|T_1| > |S_1|$ but this leads to a contradiction since the demands for objects in S_1 are "stronger" at $u_\star$ than at q and $u_\star$ would not be an equilibrium price vector More formally :

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