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On the value of randomization ☆

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Abstract

The paper identifies a necessary and sufficient condition for a deterministic local optimum to be locally improved upon by a stochastic deviation. When this condition is satisfied, a method to construct the stochastic allocations that increase the objective is provided. This technique is applied to a number of adverse selection and moral hazard problems.

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1. Introduction

In most economic models, concave objectives on convex sets lead to nonrandom choices. However asymmetric information and self-selection considerations introduce nonconvexities that

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may render randomization useful. The paper provides a necessary and sufficient condition for a deterministic optimum to be improved upon locally by a stochastic deviation. The result is applied to standard moral hazard and adverse selection problems. This result allows for a similar treatment of both models, pointing out their common underlying structure.

The paper considers an abstract optimization program, with a finite number of constraints which possibly define a nonconvex set of admissible choices in a finite dimension Euclidean space. When choices are restricted to be deterministic, a condition satisfied by a local optimum is that the second derivative of the Lagrangian be negative for all deviations in the tangent space to the active constraints. In the associated random problem, where choices may be random and the functions defining the objective and constraints are the mathematical expectations of those of the deterministic program, the set of admissible deviations is larger than in the deterministic problem. Admissible deviations in the random problem which are not allowed in the deterministic problem may be exploited to improve the objective. The main technical result of the paper is that the deterministic optimum, when associated with a regular Hessian of the Lagrangian, can locally be improved upon through a random deviation if and only if the Hessian has a positive eigenvalue. We also give a constructive method to build an improving deviation.

We first apply the above result to a multidimensional static moral hazard problem with a finite number of effort levels and outcomes. Generalizing Arnott and Stiglitz [1], we find that when the agent utility is separable in effort and wages an optimal interior deterministic contract cannot be improved upon by local randomization. This property extends to the nonseparable preferences used by Grossman and Hart [4] and also to more general nonseparable preferences which exhibit a multidimensional measure of risk aversion that increases with effort. This yields results of Bennardo and Chiappori [2] in the unidimensional case. Therefore, for a deterministic optimum to be improved upon through local random deviations, it is necessary that the agent utility be nonseparable in effort and wages and that risk aversion be nonincreasing in effort.

We then consider a general class of adverse selection models, with specialization to a taxation and a monopoly pricing setup. It is known that a random tax on low incomes may discourage the more able workers from mimicking the less able and thus alleviate the incentive constraints (Stiglitz [10]). The more general results on the usefulness of random taxation have been obtained in a two good economy by Brito, Hamilton, Slutsky, and Stiglitz [3]. Our technique provides a simple way of deriving their result and allows for a transparent economic interpretation.

Our other adverse selection application concerns discrimination strategies of a monopolist through differential risk exposure. The quality of a good often depends on future contingencies, e.g., the journey has random features associated with strikes, equipment malfunction or unreliable aircrafts. Risk averse customers facing such alternatives will usually seek some form of insurance. Firms can exploit differences in the risk aversion of their customers by a suitable design of risk exposure. Airline companies thus offer high price tickets insuring businessmen who want to be on time against random delays. Following Maskin and Riley [7], in a Mussa and Rosen [8] setup, we provide the condition under which a monopolist facing customers with different risk aversions should randomize the quality of service.

The paper is organized as follows. Section 2 presents the mathematical properties that underlie the paper. Section 3 deals with the moral hazard model. Finally Section 4 studies adverse selection, with applications to taxation and monopoly pricing.

2. A mathematical result

Consider the following constrained optimization problem:

$$\begin{cases} \max_x f(x) \\ g_n(x) \geqslant 0, \quad n = 1, \ldots, N \end{cases}$$

where x is in $\mathbb{R}^M$. The functions $f(\cdot)$ and $g_n(\cdot)$, $n = 1, \ldots, N$, are twice continuously differentiable. We do not impose convexity restrictions on the objective $f(\cdot)$, nor on the constraints $g_n(\cdot)$. We shall refer to this program as the *deterministic problem*. The associated Lagrangian $\mathcal{L}(x, \lambda)$ is the function $f(x) + \lambda' g(x)$, where λ is a vector of $\mathbb{R}^N_+$.

Some of our results require qualification of the constraints. The nth constraint is active at some point x of the domain when $g_n(x) = 0$. The constraints are qualified at x when the gradient vectors $\nabla g_n(x)$ of the active constraints at x are linearly independent.

The following property is drawn from Simon and Blume [9, Theorems 18.4 and 19.8].

Theorem 1. *Let x^* be an interior local maximum of the deterministic problem where the constraints are qualified.*

1. *There exists $\lambda^* \geqslant 0$ such that $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$, and the complementary slackness conditions $\lambda^*_n g_n(x^*) = 0$ hold for all n.*
2. *The Hessian $\nabla^2_x \mathcal{L}(x^*, \lambda^*)$ is negative semi-definite on the tangent space to the active constraints at x^*, i.e., $x' \nabla^2_x \mathcal{L}(x^*, \lambda^*) x \leqslant 0$ for all x such that $\nabla g_n(x^*)' x = 0$ for all constraints n with $g_n(x^*) = 0$.*

Let x^* in $\mathbb{R}^M$ be a point that satisfies the first order conditions given in part 1 of Theorem 1. We prove a converse to Theorem 1. Suppose that the condition given in part 2 of Theorem 1 is not satisfied in the following sense: there is a direction x^+ in the tangent space to the active constraints such that $x^{+\prime} \nabla^2_x \mathcal{L}(x^*, \lambda^*) x^+ > 0$. We are interested in the feasible deviations which improve the objective in this circumstance. A *deterministic* deviation from x^* is a continuous function $h(t)$ from [0, 1] into $\mathbb{R}^M$ such that $h(0) = 0$ and

$$g_n\big(x^* + h(t)\big) = g_n\big(x^*\big) = 0 \tag{1}$$

for all n such that $g_n(x^*) = 0$.

Theorem 2. *Let x^* be a point where the constraints are qualified and the first order conditions of Theorem 1.1 are satisfied. Suppose that there is a direction x^+ in the tangent space to the active constraints such that*

$$x^{+\prime} \nabla^2_x \mathcal{L}\big(x^*, \lambda^*\big) x^+ > 0.$$

Then there exists a deterministic deviation $h(t) = t x^+ + \beta(t)$ satisfying

$$\nabla g_n\big(x^*\big)' \beta(t) + \frac{1}{2} t^2 x^{+\prime} \nabla^2 g_n\big(x^*\big)' x^+ = o\big(t^2\big),$$

for all the active constraints n with $g_n(x^*) = 0$, such that $f(x^* + h(t)) > f(x^*)$ for small enough t different from 0.

Consider now the following maximization problem:

$$\begin{cases} \max_{\tilde{x}} \mathbb{E} f(\tilde{x}) \\ \mathbb{E} g_n(\tilde{x}) \geqslant 0, \quad n = 1, \dots, N \end{cases}$$

where $\tilde{x}$ is a random variable with values in $\mathbb{R}^M$, such that the mathematical expectations of $f(\tilde{x})$ and the $g_n(\tilde{x})$, $n = 1, \dots, N$, are well defined. We shall refer to this program as the *random problem*.

Let x^* be a point in $\mathbb{R}^M$ which satisfies the constraints. A *random* deviation $\tilde{h}(t)$ from x^* is an application from $[0, 1]$ into the random variables in $\mathbb{R}^M$ which satisfies

$$\mathbb{E} g_n\big(x^* + \tilde{h}(t)\big) = g_n\big(x^*\big) = 0 \tag{2}$$

for all active constraints, such that $\tilde{h}(0) = 0$ and the diameter of the support of $\tilde{h}(t)$ is a continuous function of t. Then we have:

Theorem 3. *Let x^* be a point where the constraints are qualified, the first order conditions of Theorem 1.1 are satisfied, and the Hessian $\nabla_x^2 \mathcal{L}(x^*, \lambda^*)$ is of full rank.*[1]

1. *If $\nabla_x^2 \mathcal{L}(x^*, \lambda^*)$ is negative definite, x^* is a local maximum of the random problem: there exists an open neighbourhood $V(x^*)$ of x^* in $\mathbb{R}^M$ such that*

   $$f\big(x^*\big) > \mathbb{E} f(\tilde{x})$$

   for all random variables $\tilde{x}$, $\tilde{x} \neq x^*$, with support contained in $V(x^*)$ such that $\mathbb{E} g_n(\tilde{x}) = g_n(x^*) = 0$ for the active constraints.
2. *Suppose that $\nabla_x^2 \mathcal{L}(x^*, \lambda^*)$ has a positive eigenvalue. To any vector x^+ such that $x^{+\prime} \nabla_x^2 \mathcal{L}(x^*, \lambda^*) x^+ > 0$, one can associate a random deviation $\tilde{h}(t)$ equal to $t x^+ + \beta(t)$ and to $-t x^+ + \beta(t)$ with equal probabilities satisfying*

   $$\nabla g_n\big(x^*\big)' \beta(t) + \frac{1}{2} t^2 x^{+\prime} \nabla^2 g_n\big(x^*\big) x^+ = o\big(t^2\big)$$

   for all the active constraints, such that $\mathbb{E} f(x^* + \tilde{h}(t)) > f(x^*)$ for small enough t different from 0.

We have therefore:

Corollary 1. *Suppose that the Hessian $\nabla_x^2 \mathcal{L}(x^*, \lambda^*)$ is of full rank. There exists a random deviation that locally improves upon the deterministic x^* if and only if this Hessian has a positive eigenvalue.*

Theorems 1 to 3 provide a detailed picture of the local properties of a solution x^* to the first order conditions of the deterministic constrained optimization problem. From Theorem 1, if x^* cannot be improved upon by deterministic moves in the constrained set, the second derivative of the Lagrangian is negative semi-definite on the tangent plane to the active constraints. Conversely, from Theorem 2, if there exists a direction in the tangent plane along which the Hessian

[1] This last assumption is discussed in Remark 1 below.

of the Lagrangian is strictly positive, x^* is not a local optimum of the deterministic problem, and there are local deterministic deviations in the constrained set which yield a higher value of the objective. Theorems 1 and 2 do not tell us anything on the behaviour of the second derivative of the Lagrangian out of the tangent plane to the active constraints, indeed a region which is forbidden territory to the deterministic problem. Our contribution in this respect is Theorem 3. It shows that any direction along which the Hessian of the Lagrangian is positive, belonging or not to the tangent plane to the active constraints, allows to build a feasible improving random deviation. If the number of active constraints is N_a, the tangent plane to the active constraints has dimension $M - N_a$ and its complement of dimension N_a is the size of the space in which the second derivative of the Lagrangian at a local deterministic optimum may be positive and generate improving random deviations.

The argument yielding Theorem 3 is constructive. Consider a local deterministic maximum x^* and small deviations $\mathrm{d}\tilde{x}$ such that all the active constraints at x^* remain binding at the new point $x^* + \mathrm{d}\tilde{x}$. The change in the objective is therefore

$$\mathbb{E}f\bigl(x^* + \mathrm{d}\tilde{x}\bigr) - f\bigl(x^*\bigr) = \tilde{\mathcal{L}} - \mathcal{L}, \tag{3}$$

where the Lagrangian $\mathcal{L}$ is evaluated at x^* and the Lagrangian $\tilde{\mathcal{L}}$ at the new point.

Since x^* satisfies the first order conditions in Theorem 1.1, the reform yields at most a second order change to the objective,

$$\tilde{\mathcal{L}} - \mathcal{L} = \frac{1}{2}\mathbb{E}(\mathrm{d}\tilde{x})' H (\mathrm{d}\tilde{x}) + o\bigl(\|\mathrm{d}\tilde{x}\|^2\bigr). \tag{4}$$

One can build deviations $\mathrm{d}\tilde{x}$ that increase the objective when H has a positive eigenvalue. The deviation involves two parts: a lottery with zero expected value in the direction of the eigenvector x^+ associated with the positive eigenvalue of H, and a deterministic part $\beta(t)$ chosen so that the binding constraints of the program at x^* stay binding along the deviation. The deviation is parameterized with a small positive scalar t which measures the scale of the change along the direction x^+. The deviation takes two values

$$\mathrm{d}\tilde{x}^1 = tx^+ + \beta(t), \qquad \mathrm{d}\tilde{x}^2 = -tx^+ + \beta(t),$$

drawn independently with equal probability. The proof of Theorem 3 shows that $\beta(t)$ is of small order in t, so that for t close enough to 0, (3) and (4) yield

$$\tilde{\mathcal{L}} - \mathcal{L} = \frac{1}{2}t^2 x^{+\prime} H x^+ + o\bigl(t^2\bigr) > 0.$$

Remark 1. Theorem 3 is shown under the assumption that the Hessian of the Lagrangian is of full rank. In practice this restriction may fail to hold in two different circumstances:

1. The Lagrangian may be linear in some directions, with all derivatives of order two and larger being equal to zero on these directions in a neighbourhood of x^*. For x^* to be the optimum, all the derivatives of the Lagrangian along these directions must be zero, and the problem is reduced to the complement directions. Generically the Hessian of the reduced system is of full rank and Theorem 3 then can be applied.
2. The Hessian is not of full rank, but the function is not locally linear, with some derivative of order larger than 2 not equal to zero in the directions along which the Hessian is null. This is a nongeneric case, which is not covered by our analysis.

Remark 2. Theorem 3 has scope provided that some of the constraints in the random problem are written in expectation. If none of the constraints is written in expectation, then randomization is not valuable.[2] Indeed, consider the (mixed) problem

$$\begin{cases} \max\limits_{\tilde{x}} \mathbb{E} f(\tilde{x}) \\ g_n(x) \geqslant 0, \quad n = 1, \dots, N \end{cases}$$

Here the N_a active constraints have to hold in all random events, so that given $M - N_a$ possibly random coordinates of $\tilde{x}$, the active constraints allow to compute locally the N_a remaining ones as a deterministic function of the latter. Now substituting into the objective yields an unconstrained optimization in $M - N_a$ unknowns. Therefore the mixed problem has the same optimum as the deterministic one.

3. Moral hazard

3.1. General setup

Consider an agent who can choose between a finite number of effort levels i $(i = 1, \dots, I)$. Effort i induces a probability distribution on the set of possible outcomes s, $s = 1, \dots, S$. The probability of seeing s after taking effort i is p_{is}, with $p_{is} \geqslant 0$ and $\sum_s p_{is} = 1$ for all i. The principal has a von Neumann–Morgenstern utility index $u_{is}(z)$ when the effort is i and the outcome is s, where z is a K-dimensional vector of transfers from the principal. In most of the literature, K equals 1, but larger Ks ease the comparison with adverse selection models. The von Neumann–Morgenstern utility index of the agent is $v_{is}(z)$. We assume that u_{is} (resp., v_{is}) is decreasing (resp., increasing) in any component z^k of z.

The outcome s is publicly observed and verifiable. The effort chosen remains private information to the agent. The principal sets a transfer schedule conditional on outcome to which she is committed. The schedule is said to be *deterministic* when each outcome s is associated with one K-dimensional vector of transfers z_s only. It is *random* when the transfers take the form of a lottery $(\tilde{z}_s)$ for some outcome s.

Assume that the principal wants the agent to undertake effort level h. The principal chooses a transfer schedule $(\tilde{z}_s)$ for every outcome s that maximizes

$$\sum_{s=1}^{S} p_{hs} \mathbb{E} u_{hs}(\tilde{z}_s)$$

subject to the incentive constraints

$$\sum_{s=1}^{S} p_{hs} \mathbb{E} v_{hs}(\tilde{z}_s) \geqslant \sum_{s=1}^{S} p_{is} \mathbb{E} v_{is}(\tilde{z}_s) \quad \text{for all } i \neq h, \qquad (\lambda_{hi})$$

and the individual rationality constraint

$$\sum_{s=1}^{S} p_{hs} \mathbb{E} v_{hs}(\tilde{z}_s) \geqslant \bar{v}, \qquad (\rho_h)$$

[2] We are grateful to a referee for pushing us to investigate this question.

where the agent's utility when he breaks his relationship with the principal for his next-best opportunity is set to $\bar{v}$.

The Lagrangian function of the deterministic problem is

$$\mathcal{L} = \sum_{s=1}^{S} p_{hs}\left[u_{hs}(z_s) + \rho\big(v_{hs}(z_s) - \bar{v}\big) + \sum_{i \neq h} \lambda_{hi}\left(v_{hs}(z_s) - \frac{p_{is}}{p_{hs}} v_{is}(z_s)\right)\right].$$

The Lagrangian is the sum over the states of functions of z_s. This separability over the states simplifies the analysis. The Hessian H of the Lagrangian associated with the deterministic optimum is a block diagonal matrix of dimension $SK \times SK$ whose sth block H_s is the $K \times K$ matrix $\nabla^2_{z_s}\mathcal{L}$ of the second derivatives of $\mathcal{L}$ with respect to z_s

$$H_s = p_{hs}\left[\nabla^2 u_{hs} + \left(\rho + \sum_{i \neq h} \lambda_{hi}\right)\nabla^2 v_{hs} - \sum_{i \neq h} \lambda_{hi} \frac{p_{is}}{p_{hs}} \nabla^2 v_{is}\right]. \tag{5}$$

The SK eigenvalues of H are the eigenvalues of the matrices H_s, $s = 1, \ldots, S$. A direct application of Theorem 3 yields

Proposition 1. *Suppose that the Hessian of the Lagrangian is of full rank at the deterministic optimum. A necessary and sufficient condition for local randomness to improve upon the optimum is that for some outcome s, one eigenvalue of the matrix H_s be positive. Then the random deviation can be supported by the corresponding eigenvector, bearing on the transfers associated with outcome s.*

In the case where the von Neumann–Morgenstern utility indices are concave, the first two terms in (5) are negative definite, whereas the last sum is positive definite. Intuitively, a random z is useful in outcome s if this outcome is likely to occur when the agent has undertaken effort i rather than the desired effort level h (p_{is}/p_{hs} is high), and noise on the transfers discourages the agent to undertake effort i (the risk aversion of the agent when effort is i is greater than in the situation where effort is h, i.e., $\nabla^2 v_{is}(z_s)$ is large in absolute value compared with $\nabla^2 v_{hs}(z_s)$).

Proposition 2. *Suppose that, at the deterministic optimum, the matrix*

$$\frac{\partial v_{is}}{\partial z^k}\nabla^2 v_{hs} - \frac{\partial v_{hs}}{\partial z^k}\nabla^2 v_{is}$$

is negative definite for all $i \neq h$ and some kth coordinate z^k of the vector of transfers. Then, local randomization of the transfers in outcome s decreases the utility of the principal.

Proof. The first order conditions in z_s are

$$p_{hs}\left[\nabla u_{hs} + \left(\rho + \sum_{i \neq h} \lambda_{hi}\right)\nabla v_{hs} - \sum_{i \neq h} \lambda_{hi} \frac{p_{is}}{p_{hs}} \nabla v_{is}\right] = 0. \tag{6}$$

The kth equation of this system can be used to get the expression of

$$\rho + \sum_{i \neq h} \lambda_{hi}.$$

Reintroducing this expression into (5) shows that H_s is negative definite, i.e., all its eigenvalues are nonpositive. This concludes the proof. □

Local randomization is therefore unprofitable in the standard specification where the utility of the agent is separable in effort, ∇v_{is} is independent of effort, in line with Arnott and Stiglitz [1].

The previous literature has usually considered the simpler case where z is a real number which represents the wage of the agent. In Grossman and Hart [4] the agent produces q_s with probability p_{is} when she undertakes effort i. The principal then gets $q_s - z_s$. The utility of the agent is $v_i(z_s) = G_i + K_i v(z_s)$, where G_i and K_i are real parameters possibly varying with effort. It is not separable in effort. For this specification, the matrix given in Proposition 2 reduces to $K_i v'(z_s) K_h v''(z_s) - K_h v'(z_s) K_i v''(z_s)$, which equals 0. Hence the Hessian is an $S \times S$ diagonal matrix with component (s, s) equal to $H_s = p_{hs} v_h''(z_s)/v_h'(z_s) < 0$. It is negative definite, and so there is no useful local randomization.

For more general preferences of the agent $v_{is}(z)$, with z a real number, Proposition 2 shows that local randomization is still useless in outcome s when the risk aversion of the agent, evaluated at the deterministic optimum, is higher when he undertakes the desired effort, i.e.,

$$-\frac{v_{hs}''(z_s)}{v_{hs}'(z_s)} > -\frac{v_{is}''(z_s)}{v_{is}'(z_s)} \quad \text{for all } i \neq h.$$

This is the condition found by Bennardo and Chiappori [2]. Assuming that the principal wants to implement the highest effort level, useful randomization requires that the absolute risk aversion decreases with effort.

4. Adverse selection

4.1. General setup

The principal faces a continuum of agents of different types i, $i = 1, \dots, I$, with whom she contracts. A deterministic contract is a K-dimensional vector z. When a type i agent chooses contract z, he gets utility $v_i(z)$ while the principal receives $u_i(z)$. The functions u_i and v_i are concave von Neumann–Morgenstern utility indices, and we allow for *ex ante* random contracts. The *ex ante* utility of a type i agent receiving a random contract $\tilde{z}$ is $\mathbb{E}v_i(\tilde{z})$, while that of the principal is $\mathbb{E}u_i(\tilde{z})$. His type is private information to the agent. The principal knows the distribution of types in the population but does not observe individual types.

Under the revelation principle, the principal chooses a menu of random contracts $(\tilde{z}_i)$, $i = 1, \dots, I$, solution to the program $\tilde{P}$

$$\max \sum_{i=1}^{I} n_i \mathbb{E}u_i(\tilde{z}_i)$$

subject to individual rationality constraints

$$\mathbb{E}v_i(\tilde{z}_i) \geqslant \bar{v}_i \quad \text{for all } i, \tag{ρ_i}$$

and incentive constraints

$$\mathbb{E}v_i(\tilde{z}_i) \geqslant \mathbb{E}v_i(\tilde{z}_j) \quad \text{for all } i \text{ and all } j. \tag{λ_{ij}}$$

The incentive constraints (λ_{ij}) make sure that when the principal announces a menu of contracts $(\tilde{z}_i)$ type i agents voluntarily choose the transfer $\tilde{z}_i$ designed for them. In the examples that we analyze below, they are crucial in generating nonconvexities, through the presence of the utility of the other agents choices on the right hand side of the constraints. On the other hand the individual rationality constraints are not essential and could be replaced with other sorts of constraints, such as feasibility requirements.

Let $\tilde{\mathcal{L}}$ be the Lagrangian function associated with this program $\tilde{P}$. From Theorem 3, the usefulness of local randomization depends on the Lagrangian $\mathcal{L}$ associated with the deterministic program denoted by P,

$$\mathcal{L} = \sum_{i=1}^{I} \left(n_i u_i(z_i) + \rho_i \big(v_i(z_i) - \bar{v}_i\big) + \sum_{j \neq i} \lambda_{ij} \big(v_i(z_i) - v_i(z_j)\big) \right). \tag{7}$$

A deterministic optimum satisfies the necessary first order conditions given in Theorem 1.1. The second order conditions in Theorem 1.2 involve the Hessian H of the Lagrangian evaluated at this point, which must be negative semi-definite on the tangent plane to the active constraints. In the current class of models the Hessian takes a specific form. Indeed H is an $IK \times IK$ symmetric matrix whose ith diagonal block is the $K \times K$ matrix

$$H_i = n_i \nabla^2 u_i(z_i) + \left(\sum_{j \neq i} \lambda_{ij} + \rho_i \right) \nabla^2 v_i(z_i) - \sum_{j \neq i} \lambda_{ji} \nabla^2 v_j(z_i) \tag{8}$$

while all off-diagonal blocks are zero.

A direct application of Theorem 3 yields

Proposition 3. *Suppose that the Hessian of the Lagrangian is of full rank at the deterministic optimum. A necessary and sufficient condition for local randomness to improve upon the optimum is that for some type i, one eigenvalue of the matrix H_i be positive. Then the random deviation can be supported by the corresponding eigenvector, bearing on the transfers designed for type i.*

The matrix H_i is negative definite when the sum of the first two terms, a negative definite matrix from the concavity of utilities, dominates the last sum which is positive definite. This happens when the profile (z_i) generates no envy. The multipliers λ_{ji} then are 0 for all $j \neq i$, so that the matrix H_i defined in (8) is negative definite. Therefore the Hessian H is negative definite.

On the other hand, suppose that a type j envies a type i at the deterministic optimum. Useful local randomization requires that type i has a lower risk aversion than type j when both take the contract designed for i. The same arguments as those used to derive Proposition 2 yield:

Proposition 4. *Suppose that, at the deterministic optimum, the matrix*

$$\frac{\partial v_j(z_i)}{\partial z_i^k} \nabla^2 v_i(z_i) - \frac{\partial v_i(z_i)}{\partial z_i^k} \nabla^2 v_j(z_i)$$

is negative definite for some kth component of z_i and for all $j \neq i$ such that $\lambda_{ji} > 0$. Then, a local randomization of the transfers (z_i) designed for agent i does not improve the objective.

The necessary condition for useful randomization given in Proposition 4 appears in Hellwig [6], where the weighted Hessian matrix is used as a multidimensional measure of risk aversion.

4.2. A taxation example

There are n_i type i agents, $i = 1, 2$, $n_1 + n_2 = 1$. Type 1 agents are ‘disabled’ and cannot supply any labour. They consume c_1 units of consumption good, yielding a utility level $u_1(c_1)$, where u_1 is increasing and concave. Type 2 agents consume c_2 and produce y_2 units of good. Their preferences are represented by $u_2(c_2) - v_2(y_2)$, with u_2 increasing and concave, $v_2(y_2)$ increasing, convex, and $v_2(0) = 0$.

Let (a_1, a_2) parameterize the redistributive tastes of the government. When type is private information, the deterministic second-best optimum (c_1^*, c_2^*, y_2^*) maximizes

$$a_1 n_1 u_1(c_1) + a_2 n_2\big[u_2(c_2) - v_2(y_2)\big]$$

subject to the feasibility constraint

$$n_1 c_1 + n_2 c_2 \leqslant n_2 y_2, \tag{9}$$

and the incentive constraint ensuring that type 2 workers must not want to fake type 1 disability,

$$u_2(c_2) - v_2(y_2) - u_2(c_1) \geqslant 0. \tag{10}$$

Disabled agents cannot work, and therefore cannot imitate the workers.

In this problem, there is a threshold $\bar{a}$ such that (10) binds for all $a_2/a_1 < \bar{a}$. For such a value of a_2/a_1, let (c_1^*, c_2^*, y_2^*) be a solution of the system formed by (9), (10) satisfied at equality, and the first order condition $u_2'(c_2) = v_2'(y_2)$. This is a local deterministic maximum.[3]

We can now apply the techniques developed in Section 2 to study whether a small random deviation from the deterministic second-best may increase the government objective. The Lagrangian of the deterministic problem is

$$\begin{aligned}\mathcal{L} = {} & a_1 n_1 u_1(c_1) + a_2 n_2\big[u_2(c_2) - v_2(y_2)\big] \\ & + \rho[n_2 y_2 - n_1 c_1 - n_2 c_2] + \lambda\big[u_2(c_2) - v_2(y_2) - u_2(c_1)\big],\end{aligned}$$

with the Hessian

$$H = \begin{pmatrix} a_1 n_1 u_1''(c_1) - \lambda u_2''(c_1) & 0 & 0 \\ 0 & (a_2 n_2 + \lambda) u_2''(c_2) & 0 \\ 0 & 0 & -(a_2 n_2 + \lambda) v_2''(y_2) \end{pmatrix}.$$

By Proposition 3 there is a profitable local random deviation if and only if H evaluated at (c_1^*, c_2^*, y_2^*) has a positive eigenvalue, i.e., $a_1 n_1 u_1''(c_1^*) - \lambda u_2''(c_1^*) > 0$.

[3] One can check that the solution to this system is a local maximum, using Theorem 1.2. The tangent plane to the active constraints is

$$\begin{pmatrix} -n_1 & -n_2 & n_2 \\ -u_2'(c_1) & u_2'(c_2) & -v_2'(y_2) \end{pmatrix} \begin{pmatrix} dc_1 \\ dc_2 \\ dy_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For all deviations in this plane, it must be that

$$(dc_1 \quad dc_2 \quad dy_2)\, H \begin{pmatrix} dc_1 \\ dc_2 \\ dy_2 \end{pmatrix} \leqslant 0.$$

From the first order conditions, there is no distortion at the top for type 2 agents, $u_2'(c_2) = v_2'(y_2)$. Therefore the only deviations (dc_1, dc_2, dy_2) from the deterministic extremum in the tangent plane to the constraints are proportional to $(0, 1, 1)$. Since the sub-Hessian H_2 corresponding to c_2 and y_2 is negative definite, any local extremum of the Lagrangian is a local maximum.

Proposition 5. *A necessary and sufficient condition for the existence of an open interval of values of social weights where the deterministic second-best optimum is locally dominated by a random allocation is*

$$\frac{r_1^A(c_1^*)}{r_2^A(c_2^*)}\left(1+\frac{n_1}{n_2}\frac{u_2'(c_2^*)}{u_2'(c_1^*)}\right)<1, \tag{11}$$

where $r_i^A(c) = -u_i''(c)/u_i'(c)$ is the coefficient of absolute risk aversion of type i when consuming c.

Proof. Eliminating the multiplier ρ between the two first order conditions $\partial\mathcal{L}/\partial c_1 = \partial\mathcal{L}/\partial c_2 = 0$ yields the multiplier λ,

$$\left[\frac{u_2'(c_1^*)}{n_1}+\frac{u_2'(c_2^*)}{n_2}\right]\lambda = a_1u_1'\big(c_1^*\big) - a_2u_2'\big(c_2^*\big).$$

The positivity of the eigenvalue is equivalent to

$$\frac{a_2}{a_1}\frac{u_2'(c_2^*)}{u_1'(c_1^*)} < 1 - \frac{r_1^A(c_1^*)}{r_2^A(c_2^*)}\left(1+\frac{n_1}{n_2}\frac{u_2'(c_2^*)}{u_2'(c_1^*)}\right),$$

which gives the desired result. Remark that in this specific example the second-best allocation (c_1^*, c_2^*, y_2^*) does not depend on the social weights (a_1, a_2), whenever it differs from the first-best. □

Since the positive eigenvalue is associated with the eigenvector with all weight on c_1, the deviation can put randomness on the disability allowance c_1 only. For randomness to be worthwhile, (11) requires that type 2 be substantially more risk averse than type 1.

4.3. Discrimination through risk exposure

The general framework used in Section 4.1 also applies to monopoly pricing, as analyzed in Mussa and Rosen [8] and Maskin and Riley [7]. The principal is a monopolist producing a commodity in different qualities. The unit cost $c(q)$ of one good of quality q is increasing and convex, with $c(0) = 0$. Each agent buys at most one good. A type i agent buying a quality q good at price p has utility $v_i(\theta_i q - p)$, with v_i increasing and concave. Tastes, represented by the function v_i and the valuation θ_i for quality, are private information. By convention valuations increase with i, $\theta_i < \theta_{i+1}$ for all $i \leqslant I - 1$.

Prices and/or quality may be random. The problem of the seller is to choose a profile $(\tilde{p}_i, \tilde{q}_i)$, $i = 1, \dots, I$, which maximizes her expected revenue

$$\sum_{i=1}^{I} n_i \mathbb{E}\big[\tilde{p}_i - c(\tilde{q}_i)\big]$$

subject to participation constraints,

$$\mathbb{E}v_i(\theta_i\tilde{q}_i - \tilde{p}_i) \geqslant 0 \quad \text{for all } i = 1, \dots, I,$$

and self-selection constraints,

$$\mathbb{E}v_i(\theta_i\tilde{q}_i - \tilde{p}_i) \geqslant \mathbb{E}v_i(\theta_i\tilde{q}_j - \tilde{p}_j) \quad \text{for all } i, j = 1, \dots, I.$$

The study of the deterministic optimum follows Maskin and Riley [7] and Guesnerie and Seade [5]. Provided that quality increases with valuation, q_i increases with i, the individual rationality constraint of type 1 consumers (associated with Lagrange multiplier λ_1) and the local neighbouring downward incentive constraints are the only relevant constraints in the nonrandom problem. Thus the Lagrangian can be written

$$\sum_{i=1}^{I} n_i\big[p_i - c(q_i)\big] + \lambda_1 v_1(\theta_1 q_1 - p_1) + \sum_{i=2}^{I} \lambda_i\big[v_i(\theta_i q_i - p_i) - v_i(\theta_i q_{i-1} - p_{i-1})\big].$$

The first order conditions yield

$$n_i c'(q_i) = N_i\theta_i - N_{i+1}\theta_{i+1}, \tag{12}$$

where N_i is the fraction of the population with valuation at least equal to θ_i (N_{I+1} is set to 0). The solution obtained from (12) must satisfy nonnegativity and the monotonicity of q_i to be economically meaningful. This is the case if the marginal cost of quality $c'(q)$ is zero at the origin and goes to infinity when q goes to infinity, $N_i\theta_i > N_{i+1}\theta_{i+1}$ for all i, and the sequence $(N_i\theta_i - N_{i+1}\theta_{i+1})/n_i$ is increasing in i. Then, the deterministic optimum is defined by a profile of qualities satisfying (12) while prices are given by the I binding constraints.

To see whether random deviations from this deterministic optimum can be profitable, we study the second derivative of the Lagrangian. Since $\lambda_i v_i' = N_i$ for all i, the Hessian of the Lagrangian is a diagonal matrix whose ith diagonal entry is

$$\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial q_i^2} &= -n_i c''(q_i) + \lambda_i\theta_i^2 v_i'' - \lambda_{i+1}\theta_{i+1}^2 v_{i+1}''\\ &= -n_i c''(q_i) - N_i\theta_i^2 r_i^A + N_{i+1}\theta_{i+1}^2 r_{i+1}^A,\end{aligned}$$

where r_i^A is the coefficient of absolute risk aversion of type i at the deterministic optimum. Theorem 3 implies:

Proposition 6. *Suppose that the Hessian is of full rank at the deterministic optimum. It is worthwhile to locally randomize the quality designed for type i consumers if and only if*

$$N_{i+1}\theta_{i+1}^2 r_{i+1}^A > N_i\theta_i^2 r_i^A + n_i c''(q_i). \tag{13}$$

Condition (13) confirms some of the intuitions seen earlier in the taxation example. From $N_{I+1} = 0$, it follows that it is never optimal to randomize the quality offered to the highest type: this comes from the fact that no other agents envy people at the top. The more convex the cost function, the higher the right hand side of (13), and the more reluctant the seller will be to randomize quality. The risk aversions of the consumers matter as expected. It is never worthwhile to randomize the quality offered to risk neutral agents.

Remark 3. In the specification used by Mussa and Rosen [8] or Maskin and Riley [7], the utility of type i consumers is separable and quasi-linear in price, $v(q_i, \theta_i) - p_i$. Then the monopolist cannot increase its expected profit with a local random deviation. To see this, recall that, by Theorem 1.2, second order conditions for a local maximum only involve deviations in the tangent space to the I binding constraints. Since the consumers' preferences are separable, these constraints allow to derive all the I expected prices as functions of the I qualities and to substitute them in the expression giving the monopolist profit. One gets an unconstrained optimization problem with respect to the qualities. It follows that at the deterministic optimum the Hessian is negative definite for all quality deviations, and by Theorem 3 local randomness cannot be profitable.[4]

Appendix A

Proof of Theorem 3. We start with part 1. The mathematical expectation of a Taylor expansion of $\mathcal{L}$ in a suitable neighbourhood of x^* is

$$\mathbb{E}\mathcal{L}(\tilde{x}, \lambda^*) = \mathcal{L}(x^*, \lambda^*) + \nabla_x \mathcal{L}(x^*, \lambda^*)(\mathbb{E}\tilde{x} - x^*) + \frac{1}{2}\mathbb{E}(\tilde{x} - x^*)' \nabla_x^2 \mathcal{L}(x^*, \lambda^*)(\tilde{x} - x^*) + o\big((\tilde{x} - x^*)'(\tilde{x} - x^*)\big).$$

From part 1 of Theorem 1, the second term on the right hand side is equal to zero. The third one is strictly negative in the chosen neighbourhood since $\nabla_x^2 \mathcal{L}(x^*, \lambda)$ is negative definite by assumption. Therefore, for $\tilde{x} \neq x^*$,

$$\mathcal{L}(x^*, \lambda^*) > \mathbb{E}\mathcal{L}(\tilde{x}, \lambda^*).$$

The active constraints at x^* are satisfied at equality, while the inactive constraints stay inactive in a suitable neighbourhood of x^*. It follows that $\lambda^{*\prime} g(x^*) = \lambda^{*\prime} g(\tilde{x}) = 0$, and consequently

$$f(x^*) > \mathbb{E}f(\tilde{x}).$$

We now prove part 2. By assumption $\nabla_x^2 \mathcal{L}(x^*, \lambda)$ has one positive eigenvalue and the associated eigenvector is a suitable x^+. *From part 2 of Theorem 1, note that x^+ cannot belong to the tangent space to the active constraints, i.e., $g_n(x^*)'x^+ \neq 0$ for some n such that $g_n(x^*) = 0$.*

By (2) the deviations $\tilde{h}(t)$ are such that $\mathbb{E}g_n(x^* + \tilde{h}(t)) = g_n(x^*)$ for the active constraints, i.e.

$$\frac{1}{2} g_n\big(x^* + tx^+ + \beta(t)\big) + \frac{1}{2} g_n\big(x^* - tx^+ + \beta(t)\big) = g_n(x^*). \tag{14}$$

We are going to show that there is a $\beta(t)$ satisfying (14) which is at most $O(t^z)$ for some $z \geqslant 2$. If there are N_a active constraints, (14) is a system of N_a equations in the unknown $\beta(t)$. A Taylor expansion of (14) yields, for all active constraints n,

$$\nabla g_n(x^*)'\beta(t) + \frac{1}{4}\big(tx^+ + \beta(t)\big)' \nabla^2 g_n(x^*)\big(tx^+ + \beta(t)\big) + \frac{1}{4}\big(-tx^+ + \beta(t)\big)' \nabla^2 g_n(x^*)\big(-tx^+ + \beta(t)\big) = o(t^2).$$

Since the constraints are qualified, the $N_a \times M$ matrix of derivatives $\nabla g_n(x^*)$ of the active constraints is of full rank.

Since by assumption the constraints are qualified, the rank of this matrix is N_a. We fix $M - N_a$ components of $\beta(t)$ at zero, so that the N_a nonzero components of $\beta(t)$ form a vector $\hat{\beta}(t)$ which

[4] This is an instance of a more general phenomenon which is at work in Strausz [11]. When some variables linearly enter the Lagrangian function, the corresponding entries of the Hessian of the Lagrangian are identically zero and the structure of the problem may simplify drastically. In the model studied by Strausz [11], there are I known active constraints with I associated transfers entering linearly both the constraints and the objective. The procedure sketched in Remark 3 then implies that there is no scope for improvement through local random deviations.

can be solved for in a neighbourhood by applying the implicit function theorem to the system made of the N_a active constraints. For each active constraint, let G_n be the $1 \times N_a$ subvector of $\nabla g_n(x^*)$ associated with the components of $\hat{\beta}(t)$. From the implicit function theorem, the function $\hat{\beta}(t)$, with $\hat{\beta}(0) = 0$, is well defined and continuously differentiable in a neighbourhood of the origin. Since $\nabla g_n(x^*)'\beta(t) = G_n\hat{\beta}(t)$, the Taylor expansion of (14) can be rewritten as

$$G_n\hat{\beta}(t) + \frac{1}{2}t^2x^{+\prime}\nabla^2 g_n\big(x^*\big)x^+ + \frac{1}{2}\beta(t)'\nabla^2 g_n\big(x^*\big)\beta(t) = o\big(t^2\big),$$

for every n in N_a. The expression in the left hand side of this equation is of smaller order than t^2 when t is in a neighbourhood of the origin since $\beta(t) \pm tx^+$ is at most $O(t)$. Stacking up these N_a equalities gives

$$G\hat{\beta}(t) + \frac{1}{2}t^2a + \frac{1}{2}b = o\big(t^2\big), \tag{15}$$

where a and b are two $N_a \times 1$ vectors,

$$a = \begin{pmatrix} \vdots \\ x^{+\prime}\nabla^2 g_n(x^*)x^+ \\ \vdots \end{pmatrix}, \qquad b = \begin{pmatrix} \vdots \\ \beta(t)'\nabla^2 g_n(x^*)\beta(t) \\ \vdots \end{pmatrix},$$

and G is the $(N_a \times N_a)$ matrix obtained by stacking up the N_a subvectors G_n of the active constraints. From the qualification of the active constraints G is of full rank and invertible. Multiplying through by the inverse of G shows that b can be neglected in (15). Indeed, t^2a is $O(t^z)$ for some $z \geqslant 2$, so that $\hat{\beta}(t)$ is at most $O(t^z)$, and thus b is at most $O(t^{2z})$. As a result, one gets

$$G\hat{\beta}(t) + \frac{1}{2}t^2a = o\big(t^2\big), \tag{16}$$

or equivalently, for all active constraints n,

$$\nabla g_n\big(x^*\big)'\beta(t) + \frac{1}{2}t^2x^{+\prime}\nabla^2 g_n\big(x^*\big)x^+ = o\big(t^2\big),$$

where the $M - N_a$ components of $\beta(t)$ associated with inactive constraints at x^* are zero, and the N_a remaining components of this vector are obtained from (16). This is the expression given in the statement of Theorem 3 that is satisfied by the deterministic component $\beta(t)$ of the deviation.

Now, by the property (14) of the random deviation,

$$\begin{aligned}
&\frac{1}{2}f\big(x^* + tx^+ + \beta(t)\big) + \frac{1}{2}f\big(x^* - tx^+ + \beta(t)\big) - f\big(x^*\big) \\
&\quad = \frac{1}{2}\mathcal{L}\big(x^* + tx^+ + \beta(t), \lambda^*\big) + \frac{1}{2}\mathcal{L}\big(x^* - tx^+ + \beta(t), \lambda^*\big) - \mathcal{L}\big(x^*, \lambda^*\big) \\
&\quad = \nabla_x\mathcal{L}\big(x^*, \lambda^*\big)'\beta(t) + \frac{1}{2}t^2x^{+\prime}\nabla_x^2\mathcal{L}\big(x^*, \lambda^*\big)x^+ + \frac{1}{2}\beta(t)'\nabla_x^2\mathcal{L}\big(x^*, \lambda^*\big)\beta(t) + o\big(t^2\big).
\end{aligned}$$

Since x^* is a local deterministic maximum, $\nabla_x\mathcal{L}(x^*, \lambda^*) = 0$. Moreover, by (16), $\beta(t) = O(t^z)$ at most, so that $\beta(t)'\nabla_x^2\mathcal{L}(x^*, \lambda^*)\beta(t)$ is at most $O(t^{2z})$. As a result,

$$\mathbb{E}f\big(x^* + \tilde{h}(t)\big) - f\big(x^*\big) = \frac{1}{2}t^2x^{+\prime}\nabla_x^2\mathcal{L}\big(x^*, \lambda^*\big)x^+ + o\big(t^2\big) > 0, \tag{17}$$

by the choice of x^+. $\square$

Proof of Theorem 2. The deviation is deterministic, $h(t) = tx^+ + \beta(t)$, and a Taylor expansion of (1) gives

$$\nabla g_n(x^*)'\beta(t) + \frac{1}{2}t^2 x^{+\prime}\nabla^2 g_n(x^*)x^+ + tx^{+\prime}\nabla^2 g_n(x^*)\beta(t) + \frac{1}{2}\beta(t)'\nabla^2 g_n(x^*)\beta(t) = o(t^2),$$

or

$$\left[\nabla g_n(x^*)' + tx^{+\prime}\nabla^2 g_n(x^*)\right]\beta(t) + \frac{1}{2}t^2 x^{+\prime}\nabla^2 g_n(x^*)x^+ + \frac{1}{2}\beta(t)'\nabla^2 g_n(x^*)\beta(t) = o(t^2).$$

From the qualification of constraints, we know that stacking up the vectors $\nabla g_n(x^*)'$ for the active constraints n gives a matrix of rank N_a. As in the proof of Theorem 3, we fix $M - N_a$ components of $\beta(t)$ at zero, and denote by $\hat{\beta}(t)$ the N_a nonzero components, chosen so that the extracted matrix is of full rank. For each active constraint, let G_n be the $1 \times N_a$ subvector of $\nabla g_n(x^*)$ associated with the components of $\hat{\beta}(t)$ and J_n the $1 \times N_a$ subvector of $x^{+\prime}\nabla^2 g_n(x^*)$ also associated with the nonzero components of β. The Taylor expansion becomes

$$[G_n + tJ_n]\hat{\beta}(t) + \frac{1}{2}t^2 x^{+\prime}\nabla^2 g_n(x^*)x^+ + \frac{1}{2}\beta(t)'\nabla^2 g_n(x^*)\beta(t) = o(t^2).$$

Let G and J be the $N_a \times N_a$ matrices obtained by stacking up the G_n and J_n over the active constraints. By the qualification of constraints, G is of full rank, so that $G + tJ$ is invertible for small enough t. The second and third terms on the left hand side are respectively $O(t^z)$ and $O(t^{2z})$ at most for all active constraints, where $z \geqslant 2$. Therefore the terms $\beta(t)'\nabla^2 g_n(x^*)\beta(t)$ are negligible, $\hat{\beta}(t)$ is at most $O(t^z)$, and the expression in the statement of the theorem holds.

Finally the expansion of $\mathcal{L}(x^* + tx^+ + \beta(t), \lambda^*) - \mathcal{L}(x^*, \lambda^*) = f(x^* + tx^+ + \beta(t)) - f(x^*)$ yields

$$\nabla_x \mathcal{L}(x^*, \lambda^*)(tx^+ + \beta(t)) + \frac{1}{2}t^2 x^{+\prime}\nabla_x^2 \mathcal{L}(x^*, \lambda^*)x^+ + o(t^2) = \frac{1}{2}t^2 x^{+\prime}\nabla_x^2 \mathcal{L}(x^*, \lambda^*)x^+ + o(t^2),$$

a positive quantity, which completes the proof. □

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