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Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand*

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Abstract

The price–quantity dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand curves is analysed. We prove that chaotic dynamical behaviour can occur, even if both the supply and demand curves are monotonic. The introduction of adaptive expectations into the cobweb model leads to price–quantity fluctuations with a smaller amplitude. However, at the same time the price–quantity cycles may become unstable and chaotic oscillations may arise. We present a geometric explanation why chaos can occur for a large class of nonlinear, monotonic supply and demand curves.

Key words: Nonlinear cobweb model; Endogenous fluctuations; Bifurcations; Chaos
JEL classification: E32

1. Introduction

The cobweb model describes the temporary equilibrium market prices in a single market with one lag in supply. The model was introduced in the thirties (for a historical account see Ezekiel, 1938) and has since then been a bench-markmodel in economic dynamics. As is well known, when suppliers have naive price expectations and bóth the supply and demand curves are monotonic, only three types of dynamics are possible: convergence to an equilibrium price, convergence to a period 2 price cycle or exploding,

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unbounded price oscillations. Recently, Artstein (1983), Jensen and Urban (1984), Lichtenberg and Ujihara (1989) and Day and Hanson (1991) have shown that, if at least one of the supply and demand curves is non-monotonic, then chaotic price fluctuations can arise.

Nerlove (1958) introduced adaptive expectations into the cobweb model with linear supply and demand curves. In this paper we investigate the dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand. We prove that chaotic price dynamics can occur generically, even if both the supply and demand curves are monotonic. The present paper contains theoretical results explaining earlier numerical results in Hommes (1991). Finkenstädt and Kuhbier (1992) present numerical evidence for the occurrence of chaos in the model for a linear supply curve and a nonlinear, decreasing demand curve. Our methods can easily be applied to prove the occurrence of chaos in that case, but in the present paper we consider the model with linear demand and nonlinear supply. In particular we investigate the model with an S-shaped, increasing supply curve. Chiarella (1988) also considered this case, but he arbitrarily approximated the expected price behaviour by the well known quadratic difference equation $x_{n+1}=\mu x_n(1-x_n)$. Unfortunately, the quadratic model is a bad approximation.[1]

2. The cobweb model

Write p_t for the price, $\hat{p}_t$ for the expected price, q_t^d for the demand for goods and q_t^s for the supply of goods, all at time t. The *cobweb model with adaptive expectations* is given by the following four equations:

$$q_t^d = D(p_t), \qquad (Demand) \tag{1}$$

$$q_t^s = S(\hat{p}_t), \qquad (Supply) \tag{2}$$

$$q_t^d = q_t^s, \qquad (Temporary\ Equilibrium) \tag{3}$$

$$\hat{p}_t = \hat{p}_{t-1} + w(p_{t-1} - \hat{p}_{t-1}), \qquad 0 \leqq w \leqq 1. \quad (Adaptive\ Expectations) \tag{4}$$

[1] As we will see below, one can not analyze the exact model by approximation with the quadratic model. One reason for this is that in the case of an S-shaped supply curve, the expected price dynamics is described by a difference equation $x_{n+1}=g(x_n)$, where g is a 1-dimensional map with two critical points (i.e. the map g has two local extrema), whereas the quadratic map has only one critical point. The dynamics generated by a map with two critical points may be quite different from the dynamics generated by a map with one critical point. In particular, as we will see, the bifurcation scenario when a parameter is varied is much more complicated.

Observe that the *adaptive expectations* in (4) can be rewritten as $\hat{p}_t = (1-w)\hat{p}_{t-1} + wp_{t-1}$, so the new expected price is a weighted average of the old expected price and the old actual price; the parameter w is called the *expectations weight factor*.

In the special case $w=1$ we get $\hat{p}_t = p_{t-1}$, so $w=1$ corresponds to the traditional cobweb model with naive price expectations. In that case the price behaviour is described by the difference equation

$$p_t = D^{-1}(S(p_{t-1})). \tag{5}$$

When supply and demand are both monotonic, then the composite map $D^{-1} \circ S$ is also monotonic, and it follows immediately that chaos can not occur. The equilibrium price p_e corresponds to the price where the supply and demand curves intersect. The equilibrium price p_e is (locally) stable if and only if

$$-1 < S'(p_e)/D'(p_e) < 1. \tag{6}$$

What happens when $0 \leq w < 1$? In the case of linear supply and demand curves, the introduction of adaptive expectations in the cobweb model reduces the price oscillations, but the equilibrium price can still be unstable. Nerlove (1958) presents data concerning the prices, the supply and the demand of cotton, wheat and corn, and concludes that in the case of wheat the equilibrium is unstable. Consequently the unstable case would appear to be empirically relevant. From now on we assume that the demand curve is decreasing and the supply curve is increasing, and that they intersect each other. Given an initial expected price $\hat{p}_0$, an initial price p_0 or an initial quantity q_0, Eqs. (1–4) uniquely determine all future expected prices, prices and quantities. From (1–3) we get $D(p_t) = S(\hat{p}_t)$, so that $p_t = D^{-1}S(\hat{p}_t)$. Substituting this last expression into (4) for time $t+1$ yields[2]

$$\hat{p}_{t+1} = (1-w)\hat{p}_t + wD^{-1}S(\hat{p}_t). \tag{7}$$

The stability condition for the unique equilibrium price p_e is

$$1 - \frac{2}{w} < S'(p_e)/D'(p_e) < 1. \tag{8}$$

This expression is less stringent than the corresponding stability condition (6)

[2] Since $p_t = D^{-1}S(\hat{p}_t)$, $q_t = S(\hat{p}_t)$, and D and S are monotonic, it follows that the price dynamics, the quantity dynamics and the dynamics of the expected prices are all qualitatively the same. Hence, the difference equation (7) completely determines the qualitative dynamics of the model. One might also write down a difference equation describing the dynamics of the prices or the quantities, but the difference equation in (7) is easier to analyse.

of the traditional cobweb model. Hence, the introduction of adaptive expectations into the cobweb model increases the possibility for *(local)* stability for given supply and demand curves. We now address the following questions: *(Q1) What can be said about the (global) price–quantity dynamics? (Q2) Is chaotic price behaviour possible, even when both the supply and demand curves are monotonic?*

3. A class of difference equations

First we investigate the class of possible difference equations $\hat{p}_{t+1}=f(\hat{p}_t)$ in (7), when both the supply and demand curves are monotonic. Let K be the following class of smooth maps $f:\mathbb{R}^+\to\mathbb{R}^+$

$$K=\{f:\mathbb{R}^+\to\mathbb{R}^+\mid f \text{ is } C^1, -\infty<f'(x)\leqq d<1, \text{for some } d>0\}. \tag{9}$$

The next theorem describes the class of difference equations generating the expected price dynamics.

Theorem 1. *(i) If the supply curve S and the demand curve D are monotonic C^1-curves, with $S'\geqq 0$ and $D'<0$, then for every w, $0<w\leqq 1$, the map f generating the expected price dynamics in (7) belongs to the class K.*

(ii) For each map f in K there exist an expectations weight factor w, $0<w\leqq 1$, a monotonic supply curve S with $S'\geqq 0$, and a monotonic demand curve D with $D'<0$, such that the difference equation $x_{n+1}=f(x_n)$ describes the dynamics of the expected prices in the corresponding cobweb model with adaptive expectations.

A proof of theorem 1 is given in the appendix. The theorem shows that there are many possibilities for the difference equation $x_{n+1}=f(x_n)$, describing the dynamics of the expected prices, in the cobweb model with adaptive expectations and monotonic supply and demand curves. The map f may be non-monotonic, with 1, 2 or more critical points. Observe however that e.g. the logistic map $g_\mu(x)=\mu x(1-x)$ does not belong to the class K, since the condition $g'_\mu(x)<1$ is not satisfied. In section 4 we present a detailed analysis of the dynamics of the expected prices in the case of an S-shaped, increasing supply curve. In that case the map f has two critical points.

4. Global dynamics

This section presents the main results concerning the global dynamics. Since we are interested in the global dynamical price behaviour, we have to be more specific about the choice of the supply and demand curves. Here we choose a linear demand curve and a general class of nonlinear, increasing, S-shaped supply curves, as described in subsection 4.1. Other choices might also be relevant and could be treated in a similar way. Some theorems concerning the dynamics are stated in subsection 4.2. In subsection 4.3 we investigate how the price dynamics change when the demand curve is shifted upwards, while subsection 4.4 concentrates on the role of the expectations weight factor. In subsection 4.5 we give a geometric explanation of the fact that chaos can occur for a large class of monotonic supply and demand curves.

4.1. A general class of S-shaped supply curves

Concerning the supply curve we start off from the following two Economic Considerations: (EC1) if prices are low, then supply increases slowly, because of start-up costs and fixed production costs; (EC2) if prices are high, then supply increases slowly, because of supply and capacity constraints.

Based on these considerations we choose a nonlinear supply curve. The simplest smooth curve satisfying (EC1) and (EC2) is an S-shaped curve S, with a unique inflection point $\bar{p}$, such that (i) the slope S' of S assumes its maximum in $\bar{p}$, (ii) S' is increasing from zero to its maximum for $p<\bar{p}$ and (iii) S' is decreasing from its maximum to zero for $p>\bar{p}$. We change coordinates and choose the inflection point of the supply curve as the new origin. Note that with respect to this new origin both 'prices' and 'quantities' can be negative; all (expected) 'prices' and 'quantities' will be bounded. We write x for the (expected) price with respect to the new origin. Let the map $g:\mathbb{R}\to\mathbb{R}$ satisfy the following conditions:

(G1) g is a bounded, increasing and continuously differentiable map.
(G2) g' has a unique maximum at $x=0$, g' is increasing for $x<0$ and g' is decreasing for $x>0$.

We assume that the supply curve S_λ is given by

$$S_\lambda(x)=g(\lambda x), \qquad \lambda>0. \tag{10}$$

The parameter λ tunes the 'steepness' of the S-shape, see Fig. 1. In all our subsequent numerical simulations we use $S_\lambda(x)=\arctan(\lambda x)$, but other choices for the S-shape yield similar results.

For simplicity we assume that the demand curve is linear and decreasing:

$$D(x)=a-bx, \qquad a\in\mathbb{R}, b>0. \tag{11}$$

By taking a linear demand curve, we are able to analyse the relationship between the nonlinearity of the supply curve and the dynamics of the model.

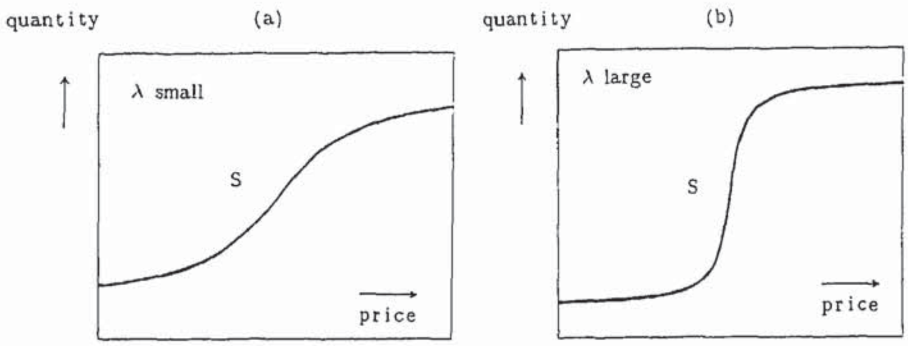

Fig. 1. The S-shaped supply curve S; the parameter λ tunes the 'steepness' of the S-shape. (a) The supply curve for λ small, (b) The supply curve for λ large.

The expected price behaviour is described by the difference equation $x_{t+1}=f(x_t)$ in (7). For the supply curve S_λ in (10) and the linear demand curve D in (11) the map f is given by

$$f_{a,b,w,\lambda}(x)=-wS_\lambda(x)/b+(1-w)x+aw/b \tag{12}$$

with parameters $a\in\mathbb{R}$, $b>0$, $0\leqq w\leqq 1$ and $\lambda>0$. Sometimes we will write f instead of $f_{a,b,w,\lambda}$. The derivative $f'_{a,b,w,\lambda}$ is

$$f'_{a,b,w,\lambda}(x)=(1-w)-wS'_\lambda(x)/b. \tag{13}$$

Chiarella (1988) also investigated the model for a linear demand curve and an S-shaped supply curve, by approximating the dynamics by the well known logistic model $x_{n+1}=g_\mu(x_n)=\mu x_n(1-x_n)$. Unfortunately, the quadratic map g_μ is a bad approximation of the map f, because (a) the map f is either increasing or f has two critical points (i.e. f has two local extrema), while the logistic map g_μ has only one critical point, and (b) the derivative of the map f satisfies $-\infty<f'(x)\leqq 1-w\leqq 1$, while the derivative g'_μ of the logistic map assumes values larger than 1 as well as values smaller than -1.

We will investigate how the dynamical behaviour of $x_{n+1}=f(x_n)$ depends on the parameters a, b, w and λ. From properties (G1) and (G2) it follows that the supply curve $S_\lambda(x)=g(\lambda x)$ has maximum slope at $x=0$. Moreover, since g is bounded, $S'_\lambda(x)$ tends to zero for $|x|$ large. If $S'_\lambda(0)\leqq b(1-w)/w$, then f is increasing, but if $S'_\lambda(0)>b(1-w)/w$, then f has two critical points c_1 and c_2, $c_1<0<c_2$, i.e. two points c_i for which $f'(c_i)=0$, $i=1,2$. Furthermore, the map f is almost linear, with slope close to $(1-w)$, for $|x|$ large. The unique fixed point x_{eq} of $f_{a,b,w,\lambda}$ is (locally) stable if and only if

$$-1<S'_\lambda(x_{eq})/b<2/w-1. \tag{14}$$

We now investigate the following question: *what can be said about the global (expected) price dynamics when the equilibrium is unstable?*

4.2. Stable 2-cycles and chaos

In this subsection we present three theorems concerning the global dynamics of the model when the equilibrium is unstable. The proofs are given in the appendix. The first result applies to the case in which the supply and demand curves intersect at the inflection point of the supply curve, that is, the case $a=0$. Theorem 2A describes a result under the extra assumption that the supply curve is symmetric, that is $S_\lambda(x)=g(\lambda x)$ is an odd function.

Theorem 2A. Assume that the supply curve S_λ in (10) is odd. If the map $f_{a,b,w,\lambda}$ in (12) satisfies $f'_{a,b,w,\lambda}(0)<-1$, then for $a=0$ $f_{a,b,w,\lambda}$ has a stable period 2 orbit, an unstable fixed point and no other periodic points.

What happens when the supply curve is not symmetric? If $f'_{a,b,w,\lambda}(0)<-1$ the map $f_{a,b,w,\lambda}$ does not necessarily have a stable period 2 orbit for $a=0$. However, in the asymmetric case, we do have a result similar to theorem 2A:

Theorem 2B. Let $f_{a,b,w,\lambda}$ be the map as given in (12) and assume that $f'_{a,b,w,\lambda}(0)<-1$. Let c_1 and c_2, $c_1<0<c_2$, be the two critical points of f. If $c_2-c_1<f(c_1)-f(c_2)$ then there exists an $\bar{a}$ such that for $a=\bar{a}$, f has a stable period 2 orbit $\{q_1,q_2\}$, with $q_1<c_1<c_2<q_2$, and f has no periodic points with period different from 1 or 2.

The condition $c_2-c_1<f(c_1)-f(c_2)$ in Theorem 2B means that the distance between the two critical points c_1 and c_2 of f is smaller than the distance between the two critical values $f(c_1)$ and $f(c_2)$. Furthermore note that the stable period 2 orbit $\{q_1,q_2\}$ lies outside the interval $[c_1,c_2]$.

Before stating the next result, we recall a definition of chaos.[3] Let $h:\mathbb{R}\to\mathbb{R}$ be a map. We say that h is a *chaotic map*, if the following three properties hold: (a) there exists a set P of infinitely many unstable periodic points with different periods, (b) there exists an uncountable set S of aperiodic points (i.e. points which are not periodic and which do not converge to a periodic point), and (c) h has sensitive dependence on initial

[3] General references, concerning chaotic systems, are e.g. Devaney (1989), Guckenheimer and Holmes (1986) and Gullick (1992). References concerning chaos and its application to economics include e.g. Lorenz (1993), Brock et al. (1992), Medio (1992) and Day (1994).

conditions with respect to the set $\Lambda = P \cup S$. The corresponding dynamical system $x_{n+1} = h(x_n)$ is called a *(topologically) chaotic dynamical system.*[4]

From a well known result by Li and Yorke (1975) it follows that if a continuous map $h: \mathbb{R} \to \mathbb{R}$ has a period 3 orbit, then h is a chaotic map. Hence, the existence of a period 3 orbit is a sufficient condition for chaos. The next theorem deals with the existence of period 3 orbits for the map $f_{a,b,w,\lambda}$.

Theorem 3. Let $f_{a,b,w,\lambda}$ be the map in (12) and assume that $0 < w < 1$ and $b > 0$. If λ is sufficiently large then there exists an interval I_1 of negative a-values and an interval I_2 of positive a-values, such that the map $f_{a,b,w,\lambda}$ has a period 3 orbit for a in I_1 and for a in I_2.

In fact, Theorem 3 says that if the S-shape of the supply curve is 'steep' (that is, if the parameter λ is large, $\lambda > M$, for some $M > 0$), then for a suitable vertical shift of the demand curve (that is, for a suitable choice of the parameter a) the dynamics is (topologically) chaotic. The value of M depends on the other parameters b and w; e.g. for $S_\lambda(x) = \arctan(\lambda x)$, $b = 0.25$ and $w = 0.3$, M is about 6.5. However, we point out that even for $\lambda < M$ chaos may occur, since the existence of a period 3 orbit is a sufficient, but not a necessary condition for the occurrence of chaos.

4.3. Shifting the demand curve: the parameter a

We now investigate how the price behaviour changes as the demand curve is shifted upwards, for different S-shapes of the supply curve. In other words, we investigate how the dynamics changes when the parameter a is increased, for different choices of the parameter λ. Note that increasing the parameter a corresponds to shifting the graph of $f_{a,b,w,\lambda}$ vertically upwards. For the numerical simulations in this subsection, we fix the slope of the demand curve $b = 0.25$ and the expectations weight factor $w = 0.3$.

When the parameter a is sufficiently large or small the slope of the supply curve $S'_\lambda(x_{eq})$, at the equilibrium x_{eq}, will be close to zero and the stability condition (14) will be satisfied. Hence, when a is small or large we have a

[4] A topologically chaotic dynamical system exhibits sensitive dependence on initial conditions with respect to the uncountable set $\Lambda = P \cup S$. Although this set is uncountable, it may have Lebesgue measure zero. In that case chaotic time paths do occur, but only with probability zero. On the other hand transient chaos occurs, that is many time paths may be influenced by the uncountable set of aperiodic points, and the (possibly long) initial part of a time path may be characterized by erratic behaviour. For more details on the notions topological chaos and sensitive dependence on initial conditions, in relation with economics, see e.g. Day and Pianigiani (1991), Grandmont (1986) and Nusse and Hommes (1990).

stable equilibrium. Suppose that the parameter λ (tuning the S-shape of the supply curve) is large enough so that when supply and demand intersect at the inflection point of the supply curve, the equilibrium is unstable. In that case, according to theorem 2A, for a symmetric supply curve like $S_\lambda(x) = \arctan(\lambda x)$ we have a stable period 2 orbit. The simplest possible bifurcation scenario with respect to the parameter a is then as shown in Fig. 2a. For $a \approx -0.9$ a *period doubling bifurcation* occurs: the equilibrium becomes unstable, and a new stable period 2 orbit is created. The stable period 2 orbit remains for an interval of a-values, containing $a=0$. For $a \approx 0.9$ a *period halving bifurcation* occurs: the stable period 2 orbit disappears and the equilibrium becomes stable again.

Figs. 2b–f show more bifurcation diagrams w.r.t. the parameter a, for increasing values of the parameter λ. For each picture the parameter a has been increased in small steps of 0.005, and for each of these a-values 300 (expected) prices were plotted after a transient time of 100 periods. For $\lambda = 3.6$ (Fig. 2b) additional period doubling and period halving bifurcations occur, involving a new stable period 4 orbit. Figs. 2a and 2b show that for small values of λ only finitely many bifurcations occur and chaos does not arise. For $\lambda = 3.9$ (Fig. 2c) and $\lambda = 4$ (Fig. 2d) the situation is much more complicated. Infinitely many period doubling as well as infinitely many period halving bifurcations occur. Chaotic price behaviour is possible for certain a-values, but in that case (indicated by arrows in Fig. 2c and 2d) the attractor is contained in 4 respectively 2 (small) disjoint intervals. A typical chaotic time path exhibits regularly alternating behaviour with respect to the unstable equilibrium, that is, in say the even time periods the value lies above x_{eq}, while in the odd periods the value lies below x_{eq}. Fig. 2e shows that for $\lambda = 4.8$ infinitely many period doubling and period halving bifurcations occur, when a is increased from -1.25 to 1.25. Chaos without regularly alternating behaviour does occur for certain values of a, indicated by arrows in Fig. 2e. In the enlargement in Fig. 2f, periodic windows can be seen, e.g. two periodic windows of period 5 are indicated by arrows. Observe that the bifurcation diagrams in 2a–e are symmetric with respect to the origin (the inflection point of the supply curve). This is of course due to the fact that the supply curve $S_\lambda(x) = \arctan(\lambda x)$ is symmetric with respect to the origin. The next theorem explains our previous numerical observations:

Theorem 4. Let $f_{a,b,w,\lambda}$ be the map in (12) and assume that $b>0$ and $0<w<1$. If λ is sufficiently large then there exist a-values $a_1 < a_2 < a_3 < a_4 < a_5$ such that the following properties hold:

(A1) *f has a globally stable fixed point, if $a \leqq a_1$.*
(A2) *the map f is chaotic for an interval of a-values containing a_2.*
(A3) *f has an unstable fixed point, a stable period 2 orbit, and no periodic*

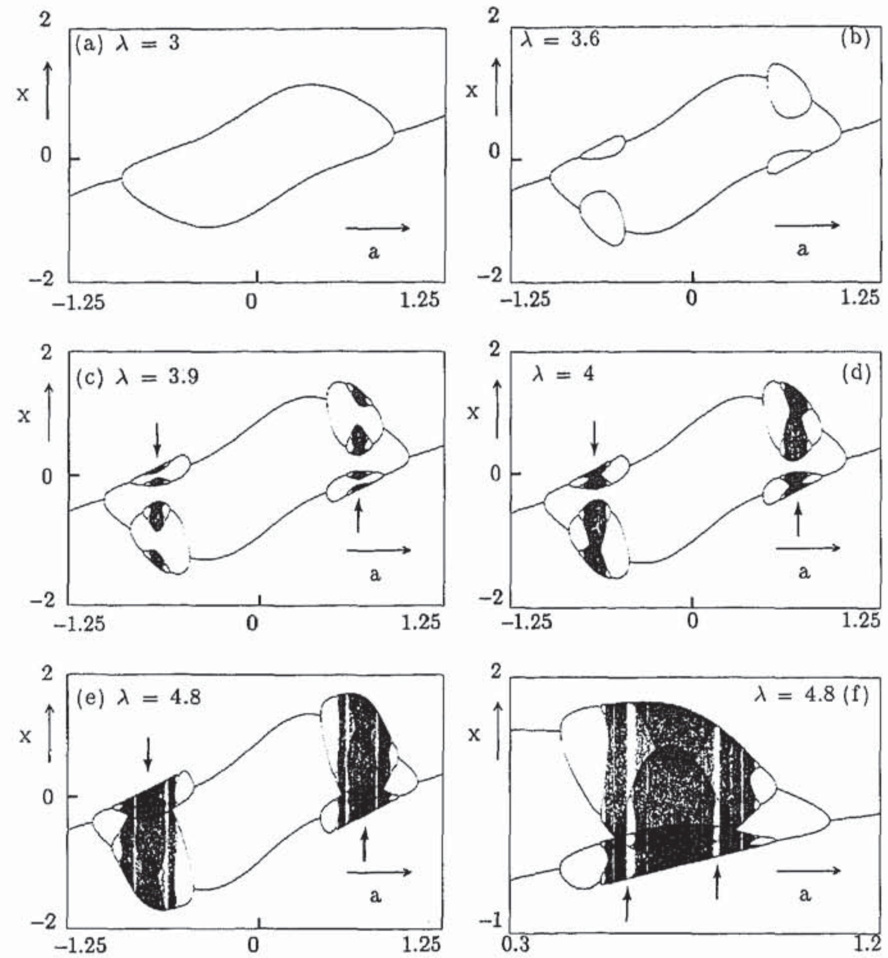

Fig. 2. Bifurcation diagrams with respect to a, $-1.25 \leqq a \leqq 1.25$, for different values of the parameter λ and with $b=0.25$ and $w=0.3$.

(a) $\lambda=3$: one period doubling and one period halving bifurcation.
(b) $\lambda=3.6$: finitely many period doubling and period halving bifurcations.
(c) $\lambda=3.9$: infinitely many period doubling and period halving bifurcations. Chaotic dynamics with 4-cyclic regularity (indicated by arrows) occurs.
(d) $\lambda=4$: infinitely many period doubling and period halving bifurcations. Chaotic dynamics with regularity alternating behaviour (indicated by arrows) occurs.
(e) $\lambda=4.8$: stable periodic and chaotic behaviour alternate several times. Chaotic dynamics without regularly alternating behaviour (indicated by arrows) occurs.
(f) $\lambda=4.8$, $0.3 \leqq a \leqq 1.2$: enlargement of (e). Stable period 5 cycles are indicated by arrows.

points with period different from 1 or 2, for an interval of a-values containing a_3.

(A4) *the map f is chaotic for an interval of a-values containing a_4.*

(A5) *f has a globally stable fixed point, if $a \geqq a_5$.*

Concerning the bifurcation scenario with respect to the parameter a, the properties (A1)–(A5) in Theorem 4 imply the following:[5]

(B1) infinitely many period doubling bifurcations occur in the parameter-intervals (a_1, a_2) and (a_3, a_4).

(B2) infinitely many period halving bifurcations occur in the parameter-intervals (a_2, a_3) and (a_4, a_5).

Theorem 4 implies that, if the 'steepness' of the S-shape of the supply curve is large, i.e. if the parameter λ is large, then the bifurcation scenario with respect to the parameter a is quite complicated and both infinitely many period doubling and infinitely many period halving bifurcations occur. Stable periodic and chaotic behaviour alternate several times.

4.4. The expectations weight factor w

In order to understand the dynamics of the nonlinear cobweb model with adaptive expectations, it is crucial to understand how the dynamics depends on the expectations weight factor w. Fig. 3 shows a bifurcation diagram w.r.t. w, $0.15 \leqq w \leqq 0.75$, with the other parameters fixed at $a=0.8$, $b=0.25$ and $\lambda=4$. A stable equilibrium price occurs for w close to 0.15. As w is increased, after infinitely many period doubling bifurcations chaotic price behaviour arises. Next, as w is further increased, after infinitely many period halving bifurcations chaos disappears and a stable period 2 cycle occurs for w close to 0.75. Roughly speaking, for w close to 0 and for w close to 1, the price behaviour is regular, while for intermediate values of w the price behaviour is irregular. If the suppliers tend to believe that today's price will also hold tomorrow (w close to 1), then the result will be a stable period 2 price cycle with large amplitude. On the other hand, if the suppliers have much more

[5] These bifurcation results can be proven by using the kneading theory, as developed by Milnor and Thurston (1988, 1977), see e.g. Whitley (1983). Nusse and Yorke (1988) present a nice example $x_{n+1}=\mu F(x_n)$ (where F is a one-hump map with negative Schwarzian derivative and μ is a parameter), for which they show that both infinitely many period doubling and infinitely many period halving bifurcations do occur, as μ is increased. However, these results can not be applied directly, since the map $f_{a,b,w,\lambda}$ has two critical points. Guckenheimer (1980) presents arguments how the kneading theory can be used to prove results concerning the bifurcation scenario for maps with more than one critical point.

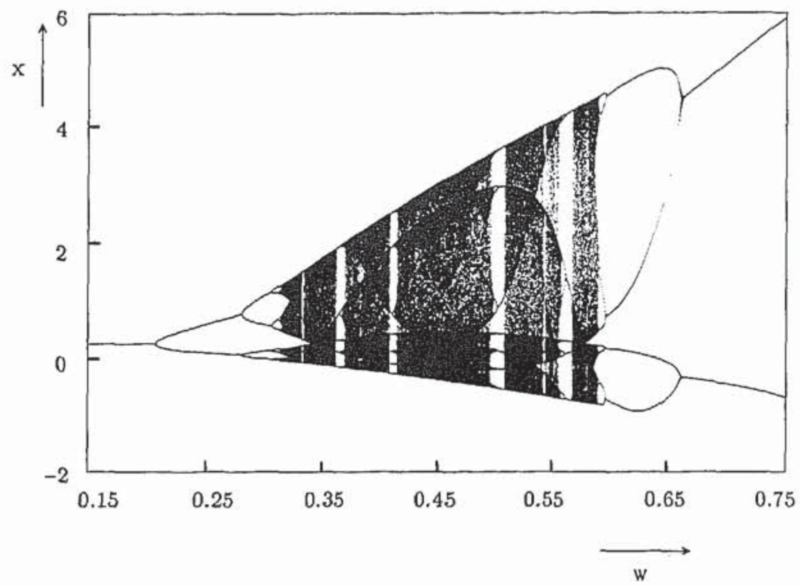

Fig. 3. Bifurcation diagram with respect to the expectations weightfactor w, $0.15 \leqq w \leqq 0.75$, with $a=0.8$, $b=0.25$ and $\lambda=4$. As w is decreased from 0.75 to 0.45 the amplitude of the price cycles becomes smaller, but at the same time the cycles become unstable and chaotic price oscillations arise. As w is decreased from 0.45 to 0.15 the price oscillations become more regular again. Both infinitely many period doubling and period halving bifurcations occur.

confidence in their own expected price than in the actual price (w close to 0) then the price behaves like a selffulfilling prophecy and will converge to a stable equilibrium price. If the suppliers hesitate between these two extreme cases (w close to say 0.5), then chaotic price behaviour is the result. Note that, as the expectations weight factor w decreases from 1 to 0, the amplitude of the price oscillations decreases. The bifurcation scenario with respect to w is explained by the following theorem:

Theorem 5. *Let $f_{a,b,w,\lambda}$ be the map in (12) and assume that $b>0$. If λ is sufficiently large then there exists an a-value a^* and w-values $w_1 < w_2 < w_3$ such that the following properties hold:*

(C1) *f has a globally stable fixed point for $0 < w < w_1$,*
(C2) *the map f is chaotic for an interval of w-values containing w_2.*
(C3) *f has a stable period 2 orbit and no periodic points with period different from 1 or 2, for $w_3 < w \leqq 1$.*

Theorem 5 implies that, if λ is sufficiently large and for a suitable choice of

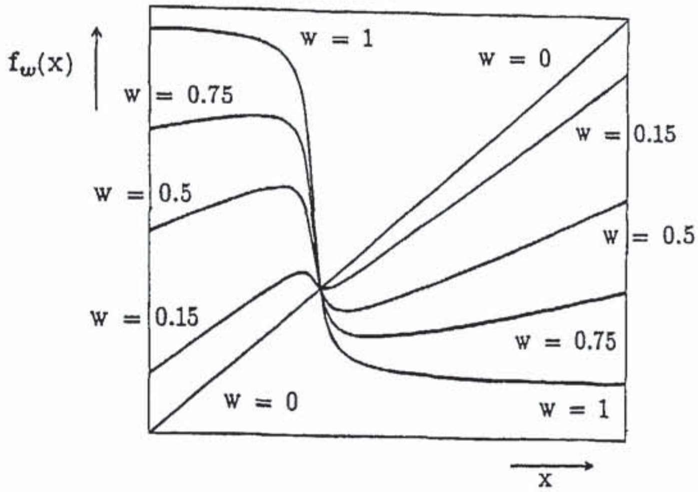

Fig. 4. Graphs of the map f_w for different values of the expectations weight factor w, with $a=0.8$, $b=0.25$, and $\lambda=4$. For w close to 0 f_w has a globally stable equilibrium. For w close to 1 f_w has a stable period 2 cycle. For w close to 0.5 the map f_w is chaotic.

the parameter a, infinitely many period doubling bifurcations occur as w is increased from 0 to w_2 and infinitely many period halving bifurcations occur, as w is increased from w_2 to 1. The parameter a has to be chosen in such a way that the supply and demand curves intersect at some 'suitable' point between the steep and the flat part of the S-shaped supply curve.

4.5. Geometric explanation of the occurrence of chaos

This subsection presents a geometric explanation how the combination of adaptive expectations and nonlinear, monotonic supply and demand curves can lead to erratic price fluctuations. First consider the case of a linear demand and an S-shaped supply curve. To stress the dependence on w, we write f_w for the map $f_{a,b,w,\lambda}$ in (12). Fig. 4 shows the graphs of f_w, for different values of w. With the parameters a, b and λ as in subsection 4.4, the maps f_w satisfy the following properties:

(W1) For $0<w<1/17$ the map f_w is increasing and has a globally stable fixed point. For $1/17<w<1$, f_w is non-monotonic and has two critical points.

(W2) For $w=1$, f_w is decreasing and has a stable period 2 orbit. For w close to 1, $w\neq 1$, f_w is non-monotonic and has a stable period 2 orbit.

(W3) All maps f_w have the same fixed point x_{eq}. The graph of f_w lies between the diagonal $y=x$ and the graph of the map $D^{-1}\circ S$. For $x<x_{eq}$, we have $x<f_w(x)<D^{-1}S(x)$, and for $x>x_{eq}$, we have $D^{-1}S(x)<f_w(x)<x$.

We first present a geometric explanation of the bifurcation scenario in Fig. 3 of subsection 4.4. Recall from (7) that for a general supply curve S and demand curve D, the map f_w is given by

$$f_w(x)=(1-w)x+wD^{-1}S(x). \tag{15}$$

For $w=0$, f_w is the identity and for $w=1$, $f_w=D^{-1}\circ S$. According to (5), the composite map $D^{-1}\circ S$ is precisely the map generating the price dynamics in the traditional cobweb model with supply curve S and demand curve D. If demand is decreasing and supply is increasing, then $D^{-1}\circ S$ is decreasing. The graph of the map f_w is a weighted average of the diagonal $y=x$ (which is increasing) and the graph of the map $D^{-1}\circ S$ (which is a decreasing curve). For our choice of supply and demand, according to property (W2), a stable period 2 price cycle occurs for w close to 1. If w is decreased, then the amplitude of the period 2 cycle will decrease, because of property (W3). Meanwhile the period 2 cycle may become unstable. Apparently, in our case, e.g. for $w=0.5$ the non-monotonic map f_w is chaotic, cf. Fig. 4. Hence, a cascade of infinitely many period doubling bifurcations occurs, as w is decreased from 1 to say 0.5. If w gets close to 0, then a globally stable equilibrium occurs, according to property (W1). Hence, as the parameter w is decreased from say 0.5 down to 0, a cascade of infinitely many period halving bifurcations occurs, cf. Fig. 3.

We emphasize that this geometric explanation is quite general. In fact the properties (W1)–(W3) (with the constant 1/17 in W1 replaced by some other constant c, $0<c<1$) hold for a large class of nonlinear supply and demand curves. The main reason for the occurrence of chaos is the following. If supply is increasing and demand is decreasing, then the graph of the map f_w is the weighted average of an increasing line (the diagonal $y=x$) and the graph of the decreasing map $D^{-1}\circ S$. If at least one of the supply or demand curves is nonlinear, then typically there is an interval of w-values for which the map f_w is non-monotonic, and for these w-values the price-dynamics can be chaotic.

In the case of linear supply and demand curves, the introduction of adaptive expectations into the cobweb model reduces the price oscillations. We now have a corresponding result in the case of nonlinear supply and demand curves. In the nonlinear case, the introduction of adaptive expectations into the cobweb model leads to price cycles with a smaller amplitude, but at the same time the cycles may become unstable and chaotic price oscillations may arise. Hence, from a quantitative point of view adaptive expectations have a stabilizing effect, but from a qualitative point of view adaptive expectations can have a destabilizing effect upon the price–quantity behaviour.

5. Concluding remarks

We have analysed the price–quantity behaviour in the cobweb model with adaptive expectations. Chaotic price–quantity dynamics can occur, even if both the supply and demand curves are monotonic. The erratic price behaviour is the result of the combination of nonlinear, monotonic supply and demand curves together with the adaptive learning of the producers.

We emphasize that our model is almost the same as the linear version of Nerlove (1958). The only difference is that we have replaced the linear supply curve by a nonlinear, increasing supply curve. In the linear case the price behaviour is always regular, while in the nonlinear case the price behaviour can be very erratic. Nonlinear supply and demand curves, together with adaptive expectations lead to price–quantity fluctuations with a smaller amplitude, but at the same time the fluctuations may become more erratic.

It is hard to believe that the cobweb model with adaptive expectations presents a realistic explanation of the price–quantity fluctuations in an independent market. The model seems to be too simple to be true. However, it is clear that nonlinear supply and demand curves are much more realistic than linear curves, and that the nonlinear version of the model has much more explaining capabilities than its linear counterpart. The nonlinear cobweb model with adaptive expectations illustrates that chaos may occur, under simple and reasonable economic assumptions. This simple example shows that it seems to be worthwhile to investigate the role of nonlinearities and the occurrence of chaos in more realistic economic models.

6. Appendix: Proofs of the results

Proof of Theorem 1. (i) The map f in (7) is $f(x)=(1-w)x+wD^{-1}S(x)$. It follows immediately that the derivative $f'(x)\leqq 1-w<1$, since $(D^{-1}S)'\leqq 0$. Therefore $f\in K$, and Theorem 1(i) follows.

(ii) Let $f\in K$. Write $g(x)=[f(x)-(1-w)x]/w$. Choose $w=1-d$, where $f'\leqq d<1$. Then $g'(x)=[f'(x)-(1-w)]/w\leqq 0$. Choose an arbitrary, C^1 demand curve D, with $D'<0$, and define $S=D\circ g$ as the supply curve. The theorem now follows immediately. □

Proof of Theorem 2A. Write f for the map $f_{a,b,w,\lambda}$ in (12), S for the supply curve S_λ in (10) and assume that $a=0$. Since the supply curve S is odd, the map f is odd. Hence, the critical points c_1 and c_2 of f satisfy $c_1=-c_2<0$ and $f(c_2)=-f(c_1)<0$. The map f has a local maximum at c_1, a local minimum at c_2, and a unique fixed point $x=0$, which is unstable. Recall that $0 \leqq f'(x) \leqq 1-w<1$, for $|x| \geqq c_2$, so $|f'(x)|$ assumes its maximum $|f'(0)|>1$ at $x=0$. We consider the cases $c_1 \leqq f(c_2)$ and $c_1 > f(c_2)$ separately.

Case 1: $c_1 \leqq f(c_2)$. First assume $c_1 < f(c_2)$. We also have $f(c_1) < c_2$, since f is odd. There exist unique points d_1 and d_2, $d_1 < c_1$ and $d_2 > c_2$, such that $f(d_1)=c_1$ and $f(d_2)=c_2$. Consequently, the critical points of the map f^2 are d_1, c_1, c_2 and d_2; f^2 has a local minimum at c_1, a local maximum at c_2, and f^2 is strictly increasing for $c_1 < x < c_2$. The slope of f^2 increases from 0 at c_1 to its maximum value $(f'(0))^2>1$ at 0, and then decreases to 0 again at c_2. Since $f^2(c_1) > f(c_2) > c_1$ and $f^2(c_2) < f(c_1) < c_2$, it follows that f^2 has two stable fixed points q_1 and q_2, $c_1 < q_1 < 0$ and $0 < q_2 < c_2$. The pair $\{q_1, q_2\}$ is a stable period 2 orbit. From the graph of f^2 one easily obtains that the set of periodic points of f consists only of the unstable fixed point and the period 2 points q_1 and q_2.

Similarly, one can show that, if $c_1 = f(c_2)$ then f has an unstable fixed point, a stable period 2 orbit $\{c_1, c_2\}$, and no other periodic points.

Case 2: $c_1 > f(c_2)$. We prove that the graph of f^2 is as illustrated in Fig. 5b, with two stable fixed points. It is sufficient to concentrate on positive x, since the map f^2 is odd, for $a=0$. The positive critical points of f^2 are c_2, the point $d_2>0$ such that $f(d_2)=c_2$ and the two positive points α and β such that $f(\alpha)=f(\beta)=c_1$. We have $0<\alpha<c_2<\beta<d_2$. The map f^2 has a local maximum at α, a local minimum at c_2, a local maximum at β and a local minimum at d_2. We claim the following properties: (i) $f^2(c_2) > c_2$, (ii) $f^2(\beta) < \beta$, (iii) $0 \leqq (f^2)'(x) \leqq (1-w)^2 < 1$, for $c_2 \leqq x \leqq \beta$.

First we show how the theorem follows from properties (i)–(iii). The slope of f^2 decreases from its maximum value $(f'(0))^2$ at 0, to 0 at $x=\alpha$. Hence, f^2 has no fixed points between 0 and α. Properties (i)–(iii) yield that f^2 has a unique stable fixed point q_1, with $c_2 \leqq q_1 \leqq \beta$. The pair $\{q_1, q_2\}$, with $q_2 = f(q_1)$, is a stable period 2 orbit. From the graph of f^2, it follows that the set of periodic points of f consists of the unstable fixed point and the period two points q_1 and q_2. Hence, it is sufficient to prove (i)–(iii).

(i) Write $c=c_2=-c_1>0$ and $d=f(c_1)=-f(c_2)>0$. Let $C_1=(-c,-c)$, $F_1=(-c,d)$, $U=(-d,-d)$ and $V=(-d, f^2(c_2))$, see Fig. 5c. Let l be the line with slope 1 through F_1. For $x<c_1$, the graph of f lies above l, since $0 \leqq f'(x) \leqq 1-w<1$. Let K and L be the points where UV intersects the x-axis and l, respectively. Since l intersects the x-axis at $(-c-d, 0)$ and the point L, has x-coordinate $-d$, it follows that the y-coordinate of L is positive. Let $\|AB\|$ denote the length of AB. We have $\|C_1F_1\| = \|UL\|$, since l is parallel to the diagonal $y=x$. Hence, $c+d=\|C_1F_1\|=\|UL\|=\|UK\|+\|KL\|$. Using $\|UK\|=d$ we get $\|KL\|=c$. We conclude that, $f^2(c_2)=\|KV\|>\|KL\|=c=c_2$.

(ii) Let $X=(c,-c)$, $F_2=(c,-d)$ and m the line with slope 1 through F_2, see Fig. 5d. For $x>c_2$, the graph of f lies below the line m. Let Y and Z

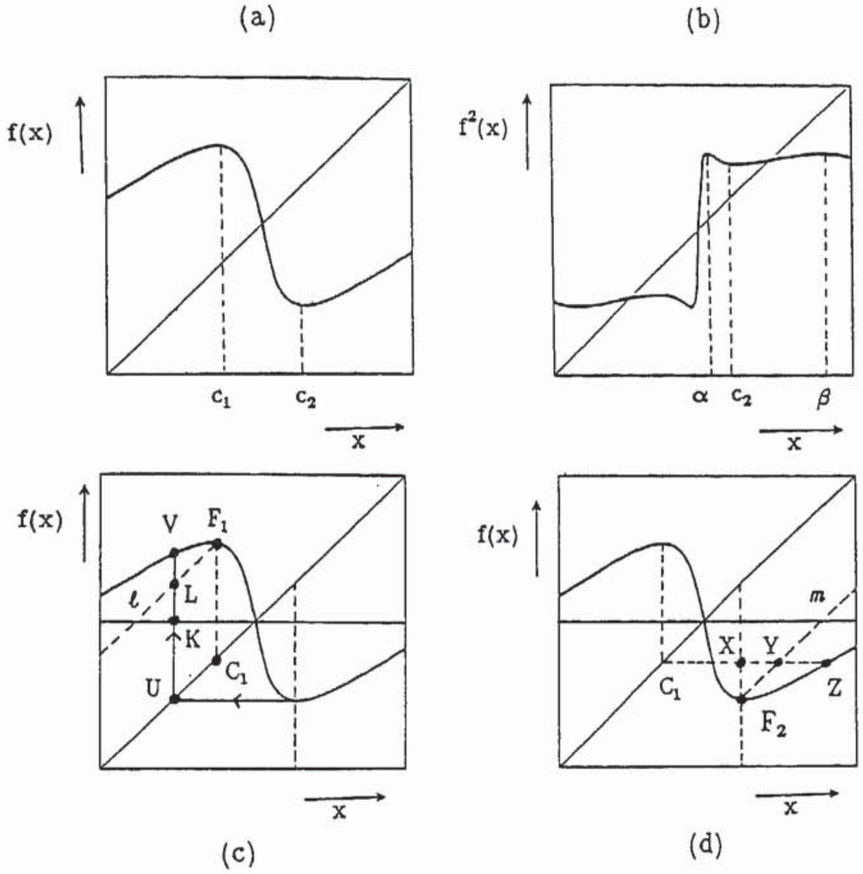

Fig. 5.
(a) The graph of the map f with the two critical points c_1 and c_2 when $c_1 > f(c_2)$.
(b) The graph of the second iterate f^2; the points α, c_2 and β are critical points of f^2.
(c) The points C_1, F_1, U, V, K and L, and the line l.
(d) The points C_1, F_2, X, Y and Z and the line m.

be the points at which C_1X intersects m and the graph of f respectively. We have $\|XF_2\| = \|XY\|$, since m has slope 1. Recall that $f(\beta) = c_1 = -c$ and $\beta > c_2$, so that $Z = (\beta, f(\beta))$. We obtain $\beta = c + \|XZ\| = c + \|XY\| + \|YZ\| = c + \|XF_2\| + \|YZ\|$. Using $c + \|XF_2\| = d = f(c_1)$ we get $\beta = f(c_1) + \|YZ\|$. We conclude that $f^2(\beta) = f(c_1) < \beta$.

(iii) Recall that $(f^2)'(x) = f'(f(x)) \cdot f'(x)$. For x between c_2 and β we have $0 \leqq f'(x) \leqq 1 - w < 1$ and $f(x) \leqq c_1$. If $f(x) \leqq c_1$ then $0 \leqq f'(f(x)) \leqq 1 - w$. Hence, $0 \leqq (f^2)'(x) \leqq (1-w)^2 < 1$, for x between c_2 and β. This completes the proof of Theorem 2A. □

Proof of Theorem 2B. Write f for the map $f_{a,b,w,\lambda}$ in (12) and $d_1=f(c_1)$ and $d_2=f(c_2)$ for the critical values of f. Let $F_1=(c_1,d_1)$ and $F_2=(c_2,d_2)$. Choose the parameter $a=\bar{a}$, such that F_1F_2 intersects the diagonal $y=x$ in the midpoint M of F_1F_2. We claim that the theorem holds, for $a=\bar{a}$. To show this, change coordinates by choosing M as the new origin. Let $\bar{c}_1$ and $\bar{c}_2$ denote the critical points, and $\bar{d}_1$ and $\bar{d}_2$ the critical values with respect to the new coordinates. Observe that $\bar{c}_1=-\bar{c}_2<\bar{0}$ and $\bar{d}_2=-\bar{d}_1<\bar{0}$. Since $\bar{c}_2-\bar{c}_1=c_2-c_1<d_1-d_2=\bar{d}_1-\bar{d}_2$, we obtain $|\bar{c}_1|<|\bar{d}_2|$ and $\bar{c}_2<\bar{d}_1$. Theorem 2b now follows by the same arguments as in case 2 of the proof of Theorem 2A. □

Proof of Theorem 3. Let $f_{a,b,w,\lambda}$ be as in (12), and let $b>0$ and $0<w<1$ be given. The supply curve $S_\lambda(x)=g(\lambda x)$, with g satisfying the assumptions (G1) and (G2) in section 4.1. Denote the lower and upper bounds of S_λ by $g_{\min}$ and $g_{\max}$, that is, $g_{\max}=\lim_{x\to\infty}g(x)$ and $g_{\min}=\lim_{x\to-\infty}g(x)$, $g_{\min}<0<g_{\max}$. Write f for $f_{a,b,w,\lambda}$ and S for S_λ. Assume that $\lambda>\lambda_0=b(1-w)/wg'(0)$, so that f has two critical points c_1 and c_2, which only depend on λ. The map f is increasing for $x<c_1$ and for $x>c_2$, while f is decreasing for $c_1<x<c_2$.

From the properties of the map g, and by using some elementary calculus the following properties of the supply curve S follow easily: For every $\delta_1>0$, and for every $\varepsilon_1>0$, there exists $M_1>0$ such that for all $\lambda>M_1$ we have:

(i) $S'(x)<\varepsilon_1$, for all x with $|x|>\delta_1$.
(ii) $S(c_2)>g_{\max}-\varepsilon_1$ and $S(c_1)<g_{\min}+\varepsilon_1$.

By using some further elementary calculus, these properties of the supply curve can be translated into the following properties of f: For every $\delta_2>0$, and for every $\varepsilon_2>0$, there exists $M_2>0$ such that for all $\lambda>M_2$ we have

(a) $|c_i|<\delta_2$, $i=1,2$ (i.e. the critical points are close to 0),
(b) $1-w-\varepsilon_2\leqq f'(x)\leqq 1-w$, if $|x|>\delta_2$. (i.e. the map f is almost linear with slope close to $1-w$, for $|x|>\delta_2$).
(c) $f(c_1)-f(c_2)>D-\varepsilon_2$, with $D=w/b\ (g_{\max}-g_{\min})$ (i.e. the distance between the critical values is close to D).

Now choose the parameter $a=\alpha_2$ (depending on λ), $\alpha_2>0$, such that $f(c_2)=c_1$, see Fig. 6 (if we choose the parameter $a=\alpha_1<0$, such that $f(c_1)=c_2$, the proof is similar). The parameter a can always be chosen in this way, since varying a is just shifting the graph of f vertically upwards or downwards. We will prove that if λ is sufficiently large, then for $a=\alpha_2$ the following inequalities hold:

$$f(c_2)<c_2<f^3(c_2)<f^2(c_2). \qquad (16)$$

It is well known that these inequalities imply the existence of a period 3 orbit, see Li and Yorke (1975). Since f depends continuously on the

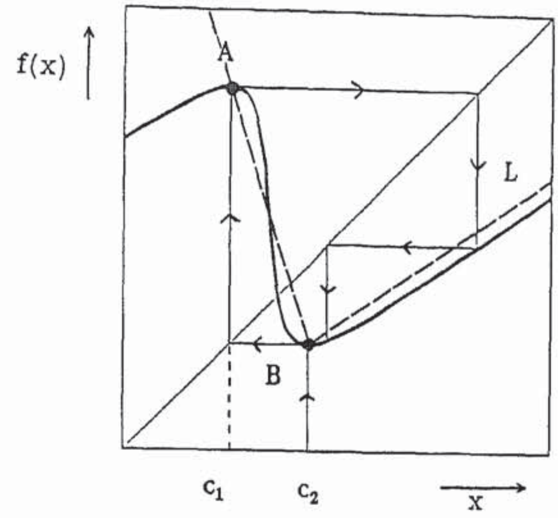

Fig. 6. The graph of the map f for $a=\alpha_2$, so that $f(c_2)=c_1$, and the graph of the piecewise linear map L in (17) (dotted lines).

parameter a it follows that if (16) holds for $a=\alpha_2$, then it holds for an interval of a-values containing $a=\alpha_2$. Hence, it is sufficient to prove (16) for $a=\alpha_2$. In order to do so we introduce a piecewise linear map L. Let $A=(c_1, f(c_1))$, $B=(c_2, f(c_2))$, $-\mu$ the slope of the line through A and B, and $\nu=1-w$, see Fig. 6. The map L is defined as:

$$L(x)=\begin{cases} -\mu(x-c_2)+f(c_2) & \text{if} \quad x \leqq c_2 \\ \nu(x-c_2)+f(c_2) & \text{if} \quad x>c_2. \end{cases} \tag{17}$$

We claim that the following properties hold:

(L1) $L(c_2)=f(c_2)$ and $L^2(c_2)=f^2(c_2)$.

(L2) there exists an $M_3>0$, such that for all $\lambda>M_3$ the following inequalities hold: $L(c_2)<c_2<L^3(c_2)<L^2(c_2)$.

(L3) for every ε, $0<\varepsilon<D(1-w)/2$, there exists an $M_4>0$ such that for all $\lambda \geqq M_4$ we have $L^3(c_2)>c_2+\varepsilon$ and $0<L^3(c_2)-f^3(c_2)<\varepsilon$.

From properties (L1)–(L3) it follows immediately that the inequalities in (16) hold, for all $\lambda \geqq M$, with $M=\max\{M_3, M_4\}$.

Property (L1) follows from the definition of L. Furthermore, observe that if λ is large then $\mu=(f(c_1)-f(c_2))/(c_2-c_1)$ is large, because of properties (a) and (c) above. In particular, $\mu>1$ for λ large. For $\mu>1$, the only nontrivial inequality in (L2) is $L^3(c_2)>c_2$. This last inequality follows also from (L3); therefore it is sufficient to prove (L3).

Let $D>0$ be defined as in property (c) above, and let $0<\varepsilon<D(1-w)/2$.

Straightforward computation shows that we have $L^3(c_2)>c_2+\varepsilon$ if and only if $\mu>1+1/v+\varepsilon/v(c_2-c_1)$. Substituting $v=1-w$ and $\mu=(f(c_1)-f(c_2))/(c_2-c_1)$ we obtain $f(c_1)-f(c_2)>(c_2-c_1)(2-w)/(1-w)+\varepsilon/(1-w)$. According to (a) and (c) above, c_2-c_1 is close to 0, while $f(c_1)-f(c_2)$ is close to D, for all $\lambda \geqq M_2$. Since $\varepsilon<D(1-w)/2$ it follows that there exists $N>0$ such that $f(c_1)-f(c_2)>(c_2-c_1)(2-w)/(1-w)+\varepsilon/(1-w)$, for all $\lambda \geqq N$. We conclude that $L^3(c_2)>c_2+\varepsilon$, for all $\lambda \geqq N$.

Finally we show that $0<L^3(c_2)-f^3(c_2)<\varepsilon$, for λ sufficiently large. We have $L^3(c_2)-f^3(c_2)>0$, since $L^2(c_2)=f^2(c_2)>c_2$ and $L(x)>f(x)$, for all $x>c_2$. Let δ_2, ε_2 and M_2 be as above. According to (a) we have $|c_i|<\delta_2$, so that $L'(x)=v=1-w$, for all $x\geqq\delta_2$. Using (b) we find that if $x\geqq\delta_2$ then $L(x)-f(x)$ increases and $L'(x)-f'(x)\leqq\varepsilon_2$. Moreover $L(\delta_2)-f(\delta_2)\leqq L(\delta_2)-f(c_2)\leqq v\delta_2=(1-w)\delta_2$ and $f^2(c_2)=f(c_1)\leqq f(c_1)-f(c_2)\leqq w/b(S(c_2)-S(c_1))\leqq w/b(g_{\max}-g_{\min})=D$. We therefore obtain $L^3(c_2)-f^3(c_2)=L(f^2(c_2))-f(f^2(c_2))\leqq L(\delta_2)-f(\delta_2)+(L'(\delta_2)-f'(\delta_2))f^2(c_2)\leqq(1-w)\delta_2+\varepsilon_2 D$. With $\delta_2=\varepsilon/2(1-w)$ and $\varepsilon_2=\varepsilon/2D$ we get $L^3(c_2)-f^3(c_2)<\varepsilon$. We conclude that property (L3) holds for all $\lambda\geqq M_4$, with $M_4=\max\{M_2,N\}$. This completes the proof of Theorem 3. □

Proof of Theorem 4. Let $b>0$ and $0<w<1$ and let S_λ be the supply curve in (10). The stability condition (14) for the equilibrium x_{eq} is given by $-1<S'_\lambda(x_{eq})/b<2/w-1$. When the parameter a increases, the equilibrium x_{eq} increases. Since the slope $S'_\lambda(x)$ of the S-shaped supply curve is close to 0 for x large as well as for x small, it follows that the equilibrium is stable when the parameter a is sufficiently large or sufficiently small. This proves properties (A1) and (A5).

Let c_1 and c_2 be the critical points of $f_{a,b,w,\lambda}$. For λ sufficiently large we have $f'_{a,b,w,\lambda}(0)<-1$ and $c_2-c_1<f(c_1)-f(c_2)$, see the proof of Theorem 3. Property (A3) now follows immediately from Theorem 2B.

From Theorem 3 it follows that, for λ sufficiently large there exist a negative a-value a_2 and a positive a-value a_4 such that $f_{a,b,w,\lambda}$ has a period 3 orbit for a-values close to a_2 and close to a_4 respectively. Properties (A2) and (A4) now follow by using the result 'Period three implies chaos' by Li and Yorke (1975). This completes the proof of Theorem 4. □

Proof of Theorem 5. Let $b>0$ be given. The derivative $f'_{a,b,w,\lambda}(x)=1-w-wS'_\lambda(x)/b$. Since S'_λ is bounded $f_{a,b,w,\lambda}$ is increasing for w close to 0. Obviously the stability condition $-1<S'_\lambda(x_{eq})/b<2/w-1$ in (14) is satisfied for w close to 0. This proves property (C1) for all values of λ and a.

By using Theorem 4 it follows that property (C2) holds for λ sufficiently large and $a=a_4$ as in Theorem 4. Now fix λ and $a=a_4$ and consider the case $w=1$. For $w=1$ $f_{a,b,w,\lambda}$ is decreasing, so that $f^2_{a,b,w,\lambda}$ is increasing. Furthermore, since the supply curve S_λ is bounded, for $w=1$ $f_{a,b,w,\lambda}$ is also bounded. From the graph of $f^2_{a,b,w,\lambda}$ it follows that $f_{a,b,w,\lambda}$ has a stable period 2 orbit and no periodic points with period different from 1 or 2. Obviously, this property will also hold for w-values close to 1. This proves property (C3) and completes the proof of Theorem 5. □

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