Eductive Stability in Real Business Cycle Models*

George W. Evans
University of Oregon
University of St. Andrews

Roger Guesnerie
Paris School of Economics
Collège de France

Bruce McGough
University of Oregon

October 23, 2013

Abstract

Within the standard RBC model, we examine issues of expectational coordination on the unique rational expectations equilibrium. Our study first provides a comprehensive assessment of the sensitivity of agents' plans and decisions to their short and long-run expectations. We show this sensitivity is much too great to trigger eductive coordination in a world of hyper-rational agents, who are endowed with Common Knowledge and contemplate the possibility of small deviations from equilibrium: eductive stability never obtains. This impossibility theorem has a counterpart when adaptive learning is incorporated and constrained to remain in line with a collective initial view of the future.

1 Introduction

The question of expectational coordination has returned to the forefront of intellectual debate in the context of the economic crisis since 2007. Some people have argued that the over-optimistic view of the world conveyed by our models arises in particular from the universal adoption of rational expectations (RE). In fact, the axiomatic use of RE has been under critical examination for some time: what is at stake in this debate is

*We are indebted for helpful comments from participants in seminars at Chicago, Columbia, the Delhi School of Economics, New York University, Oregon, and St. Andrews, and in particular from D. Pearce, H. Uhlig, T. Sadzik and M. Woodford. Financial support from National Science Foundation Grant no. SES-1025011 is gratefully acknowledged.

the robustness of coordination on RE. The issue of coordination has been discussed using a variety of approaches. Problems of multiplicity of rational expectations equilibria (REE) have received considerable attention, as in the sunspot literature, e.g. Benhabib and Farmer (1994) and Chiappori and Guesnerie (1991), and in the global games literature, e.g. Morris and Shin (1998). A second approach has examined the asymptotic stability of the REE under adaptive learning. See, for example, Marcet and Sargent (1989), Woodford (1990) and Evans and Honkapohja (2001). The central issue in this "bounded rationality" approach is whether agents can learn to form RE over time if agents are modeled as statisticians or econometricians. In the current paper we are concerned primarily with a third approach, called the "eductive" viewpoint, described below, and with its relationship to the adaptive learning viewpoint, which is examined at a later stage of the paper. See Evans and Guesnerie (1993, 2005) and Guesnerie (2002) for an introductory conceptual assessment and Guesnerie (2005) for a collection of studies along these lines.[1]

We consider here a simple Real Business Cycle (RBC) model and its standard focal point, the rational expectations equilibrium. We question the plausibility of the REE in this model, using the eductive approach. The essence of the approach is to assume full rationality of all agents and to take as given a collection of Common Knowledge assumptions. This collection includes rationality of other agents, the full structure of the model, and restrictions on the beliefs of agents concerning the path of relevant aggregates. The question, then, is whether this Common Knowledge is sufficient to trigger a mental reasoning process leading to coordination on the REE. The details of this approach are most easily introduced in the context of the cobweb model, the essentials of which are presented in Section 3.3. In the cobweb model, hyper-rational agents are able to coordinate on the REE, under natural parametric restrictions, given appropriate common knowledge assumptions.

While the introductory cobweb model argument takes place in a two-period model,[2] the logical framework and the central results for eductive learning, as well as the connections with adaptive (or "evolutive") learning, are well understood in many contexts, and in particular within simple models of the overlapping generations type in which agents are short-lived. See Gauthier and Guesnerie (2004) for an assessment that puts emphasis on the consistency of the different approaches to expectational coordination when agents have short planning horizons.[3]

In contrast, in an RBC model, agents have an infinite planning horizon. This

[1] Also relevant to this discussion is the concept of "rational beliefs" – see Kurz and Motolese (2001). Under their equilibrium concept agents have heterogeneous beliefs consistent with the empirical distribution of aggregate observables.

[2] The cobweb model is often developed as an infinitely repeated game, as in Bray and Savin (1986). There are close and well-understood connections between the eductive learning viewpoint and the standard dynamic learning viewpoint.

[3] In the context of the current paper, results for short-lived agents are given in the Appendix.

assumption plays an important role in the model under examination. In particular, long-lived agents take into account their permanent income, rather than income over a short horizon, a fact that of course has a key impact on the understanding and design of macroeconomic policies. The question under scrutiny here is the effect of the presence of long-lived agents on expectational coordination: does it make expectational coordination more or less robust?

The answer, based on our eductive assessment of coordination in this model, is that coordination of the expectations of long-lived agents is necessarily weak. There is no collective view of the future, no set of paths close but not identical to the self-fulfilling, equilibrium path, that is able to trigger coordination on the equilibrium path, along the eductive lines just described: every such collective view is subject, at some stage, to be invalidated by facts. In other words, in this simple world, a "crisis," here an expectational crisis, is unavoidable. The time required for the crisis to become manifest, i.e. the extent of weakness of expectational coordination, depends upon certain system characteristics that we later identify.

The failure of common knowledge to trigger coordination, and the real-time falsification of beliefs in the long run, suggests a central role in this setting for real-time learning. We find that the success of adaptive learning in maintaining a collective view of the future, i.e. in some sense avoiding the crisis, also depends on the system features that we stress. However, this success is again not robust, in the sense that any collective view will be falsified, at some point in time, for a large set of adaptive learning rules.

The paper proceeds as follows. In Section 2, we present the model and its equilibrium. We then provide a number of fundamental results, together with their intuition, on the connections between an agent's expectations and his optimal savings plans. In Section 3, we define eductive stability, first in the context of the cobweb model and then in a general setting. We then describe how to adapt the concept to our specific model, and distinguish between "high-tech" and "low-tech" interpretations. In Section 4 we establish our central results concerning the impossibility of strong eductive stability. Section 5 takes up adaptive learning and examines the possibility of maintaining consistency of the realized path with beliefs that the path will remain in the initially conjectured neighborhood of the equilibrium. We find that our strong eductive instability results are reflected in the paths of adaptive learning dynamics. Section 6 discusses the implications of our results and Section 7 concludes.

2 The model, equilibrium and the influence of beliefs on states

2.1 The model and equilibrium

We consider a standard RBC model, except that for simplicity we assume a fixed labor supply and omit exogenous productivity shocks.[4] These simplifying assumptions, which amount to a focus on a nonstochastic discrete-time Ramsey model, are not critical to our results and are made in order to clarify the central features of our analysis. Elimination of both random shocks and labor-supply response to disequilibrium expectations can be expected to facilitate coordination on the rational expectations equilibrium (REE).[5] Despite eliminating these influences we establish that a strong form of eductive stability fails.

2.1.1 The household problem

There is a continuum of identical infinitely-lived households, indexed by $\omega \in [0,1]$. Each household ω holds at time t, capital, $k_t(\omega)$, resulting from previous decisions, and one unit of labor, supplied inelastically. At time $t = 0$, facing interest rate r_0 and prospects of future interest rates r_t and wages q_t, household ω determines his actions today and plans for the future by solving

$$\max \quad E_0(\omega) \sum_{t=0}^{\infty} \beta^t U(c_t(\omega)), \text{ where } 0 < \beta < 1, \tag{1}$$

$$\text{subject to} \quad k_{t+1}(\omega) = (1 + r_t)k_t(\omega) + q_t - c_t(\omega), \tag{2}$$

with initial wealth $k_0(\omega)$ given. Here $E_0(\omega)$ captures the expectations of agent ω formed using his subjective distribution.

We will focus on the case in which $k_0(\omega)$ is the same for all agents, but it is convenient not to impose this initially. The utility function $U(c)$ is increasing, strictly concave and smooth. We further impose a No Ponzi Game (NPG) condition that the present value of their limiting lifetime wealth be nonnegative.

The first-order condition for the household optimization problem is the Euler equation

$$U'(c_t(\omega)) = \beta E_t(\omega) \left((1 + r_{t+1})U'(c_{t+1}(\omega))\right). \tag{3}$$

[4] The seminal papers developing the RBC model include Kydland and Prescott (1982), Long and Plosser (1983) and Prescott (1986).

[5] For example, the weak eductive stability conditions, given below, can be shown to be stricter when labor supply is elastic. Also, it is straightforward to extend our instability results to allow for *iid* random productivity shocks with small support.

When the future is viewed as deterministic, expectations of future interest rates r_t and wages q_t are point expectations. Iterating forward the household flow budget constraint, and imposing the NPG and transversality conditions, gives the lifetime budget constraint of the household:

$$\begin{aligned} \sum_{t=0}^{\infty} R_t c_t(\omega) &= \sum_{t=0}^{\infty} R_t q_t + (1+r_0)k_0(\omega), \qquad (4) \\ \text{where } R_t &= \prod_{i=1}^{t} (1+r_i)^{-1} \end{aligned}$$

and $R_0 = 1$.

2.1.2 Production and Equilibrium

Goods are produced by firms from capital and labor using a constant returns to scale production function $f(K, L)$, satisfying the usual assumptions, under conditions of perfect competition. Thus r_t and q_t are given by

$$\begin{aligned} r_t &= f_K(K_t, 1) - \delta \\ q_t &= f_L(K_t, 1), \end{aligned}$$

where $K_t = \int_0^1 k_t(\omega)d\omega$ and where $f_K = \partial f/\partial K$ and $f_L = \partial f/\partial L$ and $0 \leq \delta \leq 1$ is the depreciation rate. For convenience, below, we also write $f(K)$ in place of $f(K, 1)$ and use the notation $f' = f_K$ and $f'' = f_{KK}$. In addition we have the aggregate capital accumulation equation

$$K_{t+1} = (1-\delta)K_t + f(K_t) - C_t, \qquad (5)$$

where $C_t = \int_0^1 c_t(\omega)d\omega$.

We can now define the (unique) perfect foresight steady state.

Definition 1 *The perfect foresight steady state $K_t = k_t(\omega) = \bar{K}$, $C_t = c_t(\omega) = \bar{C}$, $r_t = \bar{r}$ and $q_t = \bar{q}$ is given by*

$$\begin{aligned} 1 &= \beta(1+\bar{r}) \\ \bar{r} &= f_K(\bar{K}, 1) - \delta \\ \bar{q} &= f_L(\bar{K}, 1) \\ \bar{C} &= f(\bar{K}, 1) - \delta\bar{K}. \end{aligned}$$

Note that agents consume what is left after depreciated capital is replaced. Also, since

$$f(\bar{K}, 1) = f_K(\bar{K}, 1)\bar{K} + \bar{q}$$

we know that $\bar{C} = \bar{r}\bar{K} + \bar{q}$ in steady state.

If $K_0 = \bar{K}$ then under perfect foresight the economy stays in the steady state for all t, and if $K_0 \neq \bar{K}$, then there is a unique perfect foresight path that converges to the steady state as $t \to \infty$. We now assume that the economy is initially in the steady state, with $k_0(\omega) = \bar{K}$ for all ω, and we examine the robustness of expectational coordination on this equilibrium.

2.2 Beliefs, actions, plans and realizations

2.2.1 Preliminaries

Consider an individual agent facing the consumption/savings problem (1)-(2). The behavior of the agent is in part determined by his beliefs about the future values of wages and interest rates. At the most general level, an agent's beliefs may be stochastic, and so summarized by a sequence of joint density functions $\{F_t(q^t, r^t)\}$, where q^t and r^t are the time t wage and interest rate histories, respectively.

Here, we shall restrict attention to deterministic beliefs, i.e. to point expectations. This is not only justified by the fact that our reference solution is nonstochastic, but also, as we will make clear later, because we focus on variations in beliefs that are small deviations from perfect-foresight beliefs.

The beliefs of agent ω may therefore be summarized by real sequences of expected wages and interest rates. We choose to assume that the agent understands the relationship between aggregate capital and input prices, that is, the agent knows

$$r_t = f'(K_t) - \delta, \text{ and } q_t = f(K_t) - f'(K_t)K_t.$$

Hence his beliefs are completely and consistently captured by a sequence of real numbers identifying his point expectations of future capital stock. We denote these beliefs by $K^e(\omega) = \{K_t^e(\omega)\}_{t\geq 0}$, where $K_0^e(\omega) = K_0 = \bar{K}$ is known to all agents. A beliefs profile is the collection of all agents' beliefs: $K^e = \{K^e(\omega) : \omega \in [0, 1]\}$.

The key ingredient of the analysis is the understanding of the effect of changes in individual expectations on changes in individual actions or plans (particularly when these changes occur around the equilibrium). Taking as reference point the perfect foresight steady-state path $K_0 = \bar{K}, K_t = \bar{K}$, for all t, we examine the solution to the program, which determines the agent's present actions and future plans. This is simply the program (1) subject to (2) and (4), where q_t and r_t are replaced by $q_t^e(\omega)$ and $r_t^e(\omega)$, derived as just explained from $K_t^e(\omega)$.

We will focus on small changes, around the steady-state values, in the individual agent's initial capital $k_0(\omega)$ and point expectations $K_t^e(\omega)$, and on the effect of these small changes on the agent's initial plans. We measure these small changes in capital as deviations from steady state: we set $dk_0(\omega) = k_0(\omega) - \bar{K}$ and $dK_t^e(\omega) = K_t^e(\omega) - \bar{K}$. We also write $dK^e(\omega) = \{dK_t^e(\omega)\}_{t\geq 0}$ for the beliefs path of agent ω, and $dK^e = \{dK^e(\omega) : \omega \in [0,1]\}$. Similarly, for the agent's corresponding optimal plans $k_t(\omega)$ and $c_t(\omega)$, which as we will see are fully determined by their beliefs and their initial capital holdings, we write $dk_t(\omega) = k_t(\omega) - \bar{K}$ and $dc_t(\omega) = c_t(\omega) - \bar{C}$. Throughout the paper we maintain the assumption that each agent's beliefs are such that their plans are well captured by first-order approximations. Given this assumption, we can identify a particular variable's time path with its first-order approximation, and thus we use our deviation notation to capture this identification.

2.2.2 Expectations and plans of agents: the preparation lemma.

Our objective is, as argued above, to determine the change in the agent's plans as functions of the changes in the agent's expectations and initial savings. We will make the connections between changes in expectations and changes in planned actions fully explicit in Lemma 4, which will be called the Preparation Lemma. However first we establish a key preliminary lemma, which will provide intuition for the results below. Here, as below, we use the first-order viewpoint, with the just introduced notation, and for this Lemma it is also useful to allow for variations in the initial aggregate capital stock, i.e. $dK_0 \neq 0$.

Lemma 1 *(Welfare Lemma). Consider any time path of beliefs $dK_t^e(\omega)$, and any initial saving $dk_0(\omega)$ and aggregate capital stock dK_0. Let $dc_t(\omega)$ be the associated sequence of consumption decisions. Then*

$$\beta^{-1} dk_0(\omega) = \sum_{t\geq 0} \beta^t dc_t(\omega). \tag{6}$$

Lemma 1, which is proven in the Appendix, relies on the fact that, given the individual budget constraint (4), the equilibrium plan $K_t = k_t(\omega) = \bar{K}$, $C_t = c_t(\omega) = \bar{C}$, remains just feasible, to first-order approximation. Indeed, as shown in the proof,

$$\sum_{t\geq 0} \beta^t dc_t(\omega) = \beta^{-1} dk_0(\omega) + \sum_{t\geq 0} \beta^t (dq_t + \bar{K} dr_t),$$

where $\sum_{t\geq 0} \beta^t (dq_t + \bar{K} dr_t)$ is the income effect of a change in current and expected prices due to a change in current and expected aggregate capital stocks, i.e. the present value of the change in wage and rental income. However, constant returns to scale technology implies $\bar{K} f_{KK}(\bar{K}, 1) + f_{KL}(\bar{K}, 1) = 0$, so that $dq_t + \bar{K} dr_t = 0$, from which the result follows. In summary, to first order, changes in current and expected aggregate capital stock have zero income effect.

Lemma 1 implies in particular that, given beliefs $dK_t^e(\omega)$, if $k_0(\omega) = \bar{K}$ then the optimal consumption path satisfies $\sum \beta^t dc_t(\omega) = 0$. Thus a change in the expected path of aggregate capital, and the corresponding expected price changes that it triggers, have no first-order impact on welfare, since it leaves the present value of consumption unchanged. This is the reason we call the above result the Welfare Lemma.[6]

To complete the intuitive picture we introduce the next Lemma. The flow budget constraint (2) leads to (7) and the Euler equation of the household (3) leads to (8). See the Appendix for details.

Lemma 2 *The consumption and savings paths for agent ω satisfy*

$$dk_{t+1}(\omega) = \beta^{-1}dk_t(\omega) - dc_t(\omega) \tag{7}$$

$$dc_t(\omega) = dc_{t-1}(\omega) + \left(\frac{\beta\bar{C}f''}{\sigma}\right)dK_t^e(\omega) \tag{8}$$

where $\sigma = -\bar{C}U''(\bar{C})/U'(\bar{C})$ is the consumption elasticity of marginal utility and where f'' denotes $f''(\bar{K})$.

We can now present the two main lemmas. Here and in the sequel, the parameter

$$\xi = -\frac{\beta^2\bar{C}}{\sigma(1-\beta)}f'' > 0,$$

which we call the "expectations feedback parameter," will be of considerable interest: it plays a central role in determining the responsiveness of consumption and savings plans to changes in expectations of the aggregate capital path.

Lemma 3 *Given beliefs path $dK^e(\omega)$, the optimal plans for agent ω satisfy*

$$dc_t(\omega) = \left(\frac{1-\beta}{\beta}\right)dk_t(\omega) + \frac{(1-\beta)\xi}{\beta}\sum_{n\geq 1}\beta^n dK_{t+n}^e(\omega) \tag{9}$$

$$dk_{t+1}(\omega) = dk_t(\omega) - \frac{(1-\beta)\xi}{\beta}\sum_{n\geq 1}\beta^n dK_{t+n}^e(\omega), \tag{10}$$

at each time.

[6] Using a similar first-order approximation argument on welfare, one can also prove that the same holds at time t i.e.

$$\beta^{-1}dk_t(\omega) = \sum_{s\geq t}\beta^s dc_s(\omega).$$

This is in fact what is shown in the proof in the Appendix.

Equations (9)-(10) can be interpreted as the time t consumption and saving functions of agent ω, since they relate the agent's consumption and saving at t to $dk_t(\omega)$, its wealth at time t, and to expected future prices, captured by $dK^e_{t+n}(\omega)$. These equations motivate our interpretation of ξ as an expectations feedback parameter: ξ measures the impact on consumption and savings of a permanent unit increase in expected future aggregate capital.

The next Lemma gives the optimal plans explicitly in terms of the full belief path $dK^e(\omega)$.

Lemma 4 *(Preparation Lemma). Assume $dk_0(\omega) = 0$. Given beliefs path $dK^e(\omega)$, the optimal plans for agent ω are given by*

$$\begin{aligned} dc_t(\omega) &= -\xi\left(\frac{1-\beta}{\beta}\right)\sum_{T\geq 1}\theta^c(t,T)dK^e_T(\omega), \ \textit{for } t\geq 0, \textit{ and} \\ dk_t(\omega) &= -\xi\sum_{T\geq 1}\theta^k(t,T)dK^e_T(\omega), \ \textit{for } t\geq 1, \textit{where} \end{aligned}$$

$$\begin{aligned} \theta^c(t,T) &= \begin{cases} -\beta^T & \textit{if} \quad 0\leq t<T \\ 1-\beta^T & \textit{if} \quad t\geq T \end{cases}, \\ \theta^k(t,T) &= \begin{cases} \beta^{T-t}-\beta^T & \textit{if} \quad 1\leq t<T \\ 1-\beta^T & \textit{if} \quad t\geq T \end{cases}. \end{aligned}$$

We can also write this as

$$\begin{aligned} dc(\omega) &= \Gamma^c\left(dK^e(\omega)\right) \equiv -\xi\left(\frac{1-\beta}{\beta}\right)\Theta^c dK^e(\omega) \\ dk(\omega) &= \Gamma^k\left(dK^e(\omega)\right) \equiv -\xi\Theta^k dK^e(\omega), \end{aligned}$$

where Θ^c and Θ^k are semi-infinite matrices. Note that the first row of Θ^c determines the first coordinate of $dc(\omega)$, i.e. $dc_0(\omega)$, resulting from $dK^e(\omega)$. Thus the (i,j) element of Θ^c corresponds to $\theta^c(i-1,j)$, and

$$\Theta^c = \begin{pmatrix} -\beta & -\beta^2 & -\beta^3 & \cdots \\ 1-\beta & -\beta^2 & -\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & -\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & 1-\beta^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Analogously the first row of Θ^k determines the first coordinate of $dk(\omega)$, i.e. $dk_1(\omega)$, resulting from $dK^e(\omega)$. Thus the (i,j) element of Θ^k corresponds to $\theta^k(i,j)$, and

$$\Theta^k = \begin{pmatrix} 1-\beta & \beta-\beta^2 & \beta^2-\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & \beta-\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & 1-\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & 1-\beta^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A proof of Lemma 4 is given in the Appendix. Intuition for the Lemma and an interpretation of the matrices obtains by noting that the T^{th} column determines the impact on the paths of consumption and savings plans of a unit increase in $dK_T^e(\omega)$. Thus consider a thought experiment in which $dK_T^e(\omega) > 0$ and $dK_t^e(\omega) = 0, \forall t \neq T$. Our aim is to derive column T of the matrix Γ^c. We do so using the Euler equation (8) and the Welfare Lemma (6). For this path $dK^e(\omega) = \{dK_t^e(\omega)\}_{t\geq 1}$, the expected interest rate remains constant at its steady state value in all periods, except for period T in which it changes by $f'' dK_T^e(\omega) < 0$. By the Euler equation planned consumption remains constant before and after the period of expected change; also, because $dK_T^e(\omega) > 0$ implies a lower expected interest rate at $t = T$, the price of time $T-1$ consumption is low relative to the price of time T consumption. These observations coupled with (8) may be summarized as follows:

$$\begin{aligned} d\check{c} &\equiv dc_0(\omega) = \cdots = dc_{T-1}(\omega) = dc_T(\omega) - \beta \bar{C} \sigma^{-1} f'' dK_T^e(\omega) \\ &> dc_T(\omega) = dc_{T+1}(\omega) = \cdots \equiv d\hat{c}. \end{aligned}$$

The Welfare Lemma states that the present value of consumption must be zero. Thus

$$\begin{aligned} 0 &= \sum_{t=0}^{T-1} \beta^t dc_t(\omega) + \sum_{t=T}^{\infty} \beta^t dc_t(\omega) = \left(\frac{1}{1-\beta}\right)\left((1-\beta^T) d\check{c} + \beta^T d\hat{c}\right) \\ &= \left(\frac{1}{1-\beta}\right)\left(d\hat{c} - \left(1-\beta^T\right) \beta \bar{C} \sigma^{-1} f'' dK_T^e(\omega)\right). \end{aligned}$$

It follows that $d\hat{c} = -\xi \left(\frac{1-\beta}{\beta}\right)\left(1-\beta^T\right) dK_T^e(\omega)$, as is stated in Lemma 4: see $\theta^c(t,T)$ for $t \geq T$. The complementary expression, $d\check{c} = -\xi\left(\beta^{T-1} - \beta^T\right) dK_T^e(\omega)$, immediately follows from the above equalities.

The next question is how to translate the consumption change to a savings change. Continuing with the same thought experiment, and using the flow budget equation (7), we get

$$dk_1(\omega) = -d\check{c} = -\xi\left(\beta^{T-1} - \beta^T\right) dK_T^e(\omega).$$

Recursively substituting into the flow budget constraint gives

$$dk_t(\omega) = -\xi(\beta^{T-t} - \beta^T) dK_T^e(\omega) \text{ for } t \leq T.$$

See $\theta^k(t, T)$ for $t \leq T$. Since consumption is constant for $t \geq T$, the Welfare Lemma applied to time T (see footnote following Lemma 1) implies that

$$d\hat{c} = (\beta^{-1} - 1)dk_T(\omega) = \bar{r}dk_T(\omega),$$

and thus agents consume the interest on their savings. It follows that, for $n \geq 1$

$$dk_{T+n}(\omega) = \beta^{-1}dk_T(\omega) - d\hat{c} = dk_T(\omega) = -\xi(1 - \beta^T)dK_T^e(\omega),$$

as indicated by Lemma 4: see $\theta^k(t, T)$ for $t > T$.

Note that the impact on $dk_t(\omega)$ is negative for all $t \geq 1$ even though the impact on $dc_t(\omega)$ changes sign over time. We emphasize this point in the next section.

2.2.3 Expectations and plans of agents: first implications of the Preparation Lemma

The Preparation Lemma, through the expectations feedback parameter ξ and the matrices Θ^c and Θ^k, fully describe the connection between expectations and the planned actions of the individual agents. We will later exploit this knowledge to obtain consequences of these beliefs for describing the possible corresponding paths of aggregate capital. At this stage we provide intuition for later results, limiting attention to discussion of the effects of beliefs on the plans of individual agents.

First we note that the *general sensitivity* of decisions to expectations is governed by the value of ξ. A high ξ means that the individual decisions and plans react strongly to expectations. We discuss later the interpretation of ξ, but it is intuitively clear that a high f'' or a low σ increases expectational sensitivity. In addition to this general expectational sensitivity, our results also stress *the profile effect* of beliefs, which are fully captured by the above matrices. Taken together, we get rich information – indeed exhaustive to first-order approximation – giving the effects of beliefs on the agent's plans.

This information can be used to provide a more comprehensive intuition about agents' plans (for example, the reader is invited to compute the sum of the rows of each matrix, in order to evaluate the effect of extreme beliefs). Here, we pick out two insights, Corollaries 1 and 2, that are particularly relevant for what follows.

First, as foreshadowed above, a striking and immediate implication of the inspection of the matrix Θ^k is the following corollary on strategic substitutability.

Corollary 1 *The map Γ^k exhibits local strategic substitutability in the sense that*

$$\frac{\partial\left(dk_t(\omega)\right)}{\partial\left(dK_T^e(\omega)\right)} < 0 \text{ for all } t, T \geq 1.$$

The connection between the infinite path of beliefs and the individual savings decision is further clarified by the following example. Suppose that expected capital is above the steady state by a fixed amount $e > 0$ for $N > 0$ periods before reverting to the steady state. That is, suppose the agent ω's beliefs profile, $dK^e(\omega)$, satisfies

$$dK_T^e(\omega) = \begin{cases} e & \text{for} \quad T = 1, \dots, N \\ 0 & \text{for} \quad T > N \end{cases} . \tag{11}$$

Because of its later importance, we call these "N-period deviation" beliefs. By acting on $dK^e(\omega)$, the operators Γ^k and Γ^c yield the savings and consumption plans of agent ω, respectively: $dk(\omega) = \Gamma^k\left(dK^e(\omega)\right)$ and $dc(\omega) = \Gamma^c\left(dK^e(\omega)\right)$.

Agent ω believes the real interest rate will be at its steady-state value after period N; thus he will hold consumption and savings constant from period N onward. These constant values can be determined using the operators Γ^k and Γ^c. The Appendix shows that we obtain:

Corollary 2 *Suppose an agent has N-period deviation beliefs (11). Then*

$$dk_t(\omega) = \begin{cases} -\xi\left(t - \bar{r}^{-1}(1-\beta^t)\beta^{N-t}\right)e & \textit{for } 1 \le t < N \\ -\xi\left(N - \bar{r}^{-1}(1-\beta^N)\right)e & \textit{for } t \ge N \end{cases} \tag{12}$$

$$dc_t(\omega) = \begin{cases} \xi\left(1 - \beta^N - \bar{r}t\right)e & \textit{for } 0 \le t < N \\ \bar{r}dk_t(\omega) & \textit{for } t \ge N \end{cases} . \tag{13}$$

Note that for $e > 0$ it follows that $dc_t(\omega) > 0$, that $dc_t(\omega)$ declines over time for $t < N$, and that $dc_t(\omega) < 0$ for $t \ge N$. For $t \ge N$ equation (13) is familiar: agent ω consumes the net return to savings, leaving the savings stock unchanged. Thus the expectation of a period of low interest rates associated with $dK_T^e(\omega) = e > 0$ for $T \le N$ leads agents to shift consumption from the future to the present. However, for our purposes equation (12) is more significant. Most strikingly, for $t \ge N$ the right-hand side implies that $dk_t(\omega)$ becomes large (in magnitude) as N gets large. Thus even if agent ω thinks that capital deviates from steady state by only some small e, and only for some finite period of time, his savings path will move, and remain arbitrarily far from steady state, provided that period is long enough.

With the operators Γ^k and Γ^c in hand we can now turn to our central concern, which is the possibility of rational agents coordinating on the rational expectations steady-state path.

3 The robustness of expectational coordination: eductive stability criteria

We first provide a definition of eductive stability based on rather abstract game-theoretical considerations. This will be the "high-tech" view on expectational coordination. However we will later see how the sophisticated high-tech viewpoint has a more intuitive counterpart that may be termed “low-tech”. Our analysis initially will leave in the shadow the time dimension of the problem. After illustrating the concepts in a simple cobweb framework, we then reintroduce time and adapt the general ideas to our infinite horizon setting.

3.1 Local eductive stability: the high-tech view

We begin by considering an abstract economy populated with rational economic agents (in all the following, we shall assume that these agents are infinitesimal, with the collection modelled as a continuum). The agents know the logic of the collective economic interactions (the underlying model). Both the rationality of the agents and the model are Common Knowledge (CK). The state of the system is denoted E and belongs to some subset $\mathcal{E}$ of some vector space.[7]

Emphasizing the expectational aspects of the problem, we view an equilibrium of the system as a state E^* such that if everybody believes that it prevails, it does prevail.[8]

Under eductive learning, as described below, each agent contemplates the possible states of the economy implied by the beliefs and associated actions of the economy’s agents. Coordination on a particular equilibrium outcome obtains when this contemplation, together with the knowledge that all agents are engaged in the same contemplation, rules out all potential economic outcomes except the equilibrium. If coordination on an equilibrium is implied by the eductive learning process, then we say that the equilibrium is eductively stable.[9] The argument can be either global or local. We now introduce the local version of eductive stability. As we will see, even local stability is at issue in our infinite-horizon model.

Formally, we say that E^* is locally eductively stable if and only if one can find some (non-trivial) small neighborhood of E^*, $V(E^*)$, such that Assertion A implies Assertion B:

Assertion A : It is “hypothetically” CK that $E \in V(E^*)$.

Assertion B : It is CK that $E = E^*$.

[7]Note that E can be a number (the value of an equilibrium price or a growth rate), a vector (of equilibrium prices), a function (an equilibrium demand function), an infinite trajectory of states, as will be the case in this paper, or a probability distribution.

[8]Note that E^* is such that the assertion “it is CK that $E = E^*$” is meaningful.

[9]The term “strongly rational” is also used. For details on the game theoretic background of our investigation see Guesnerie and Jara-Moroni (2011). A study within a “normal form” framework echoing the preoccupations of the present paper can be found in Matsui and Oyama (2006). We also remark that we here view eductive stability as a zero-one criterion. Less stringent indices of stability could also be developed, e.g. see Desgranges and Ghosal (2010) for one such approach.

Assertion A is at this stage hypothetical.[10] In the stable case the mental process that leads from Assertion A to Assertion B is the following:

1. Because everybody knows that $E \in V(E^*)$, everybody knows that everybody limits their responses to actions that are best responses to some probability distributions over $V(E^*)$. It follows that everybody knows that the state of the system will be in a set $\mathcal{E}(1) \subset \mathcal{E}$.
2. If $\mathcal{E}(1)$ is a proper subset of $V(E^*)$, the mental process goes on as in step 1, but based now on $\mathcal{E}(1)$ instead of $V(E^*)$. In this case it follows that everybody knows that the state of the system will be in a set $\mathcal{E}(2) \subset \mathcal{E}$.
3. The process continues inductively provided that at each stage, $\mathcal{E}(n)$ is a proper subset of $\mathcal{E}(n-1)$.

In the stable case, we then have a decreasing sequence $V(E^*) \supset \mathcal{E}(1) \supset \cdots \supset \mathcal{E}(n-1) \supset \mathcal{E}(n)$.[11] When the sequence converges to E^*, the equilibrium is locally eductively stable. Here "locally" refers to the fact that the initial neighborhood is small.[12] Note also that intuitively, in the small neighborhood case, whenever the first step conclusion obtains then the next step normally follows.[13]

Although inspired by game-theoretic considerations,[14] we view eductive stability as a natural expectational coordination criterion that has independent merit. The connections of the approach with the game theoretical viewpoint, for example the relationship between eductive stability and uniqueness of rationalizable strategies, is assessed in Guesnerie, Jara-Moroni (2011).[15]

---

[10] Although it might be sustained by some policy commitment.

[11] A given model or economic environment may be naturally allied with several distinct common knowledge assumptions: these common knowledge assumptions will all impose the recursive reasoning process described above, but will differ in the initial restrictions on agents' beliefs; and because of the central role the initial restriction plays in the eductive learning process, different common knowledge assumptions may produce different stability results.

[12] If the initial neighbourhood were equal to the whole space $\mathcal{E}$, then the word global would replace the word local.

[13] At step 1, $\mathcal{E}(1) \subset V(E^*) \subset \mathcal{E}$ is CK and the mental process goes on in step 2, so that the first step contraction still acts with a decreased support. See footnote 19, below, and Guesnerie, Jara-Moroni (2011) for further details.

[14] See the literature on rationalizable beliefs, e.g. Bernheim (1984) and Pearce (1984).

[15] Most of their analysis is conducted in a finite-dimensional state space $\mathcal{E}$. Thus their results would require an extension to an infinte-dimensional setting to ensure applicability to our RBC framework.

3.2 Local eductive stability: the low-tech interpretation

The above definition, based on the successful deletion of non-best responses and starting under the assumption that the state of the system is close to the equilibrium state, reflects the local version of a "hyper-rationality" viewpoint. Another plausible intuitive definition of local expectational stability would be the following: there exists a non-trivial neighborhood of the equilibrium such that, if everybody believes that the state of the system is in this neighborhood, it is necessarily the case, whatever the specific form taken by everybody's belief, that the state is in the given neighborhood.[16] With the above formal apparatus, the definition would be: one can find some non-trivial small neighborhood of E^*, $V(E^*)$, such that if everybody believes that $E \in V(E^*)$, then the state of the system will be in $\mathcal{E}(1)$, a proper subset of $V(E^*)$. Again, $V(E^*)$ is an initial belief assumption, a universally shared conjecture on the set of possible states, and for low-tech eductive stability we require that this belief cannot be falsified by actual outcomes resulting from individual actions that are best responses to some probability distributions over $V(E^*)$.[17] The argument is here low-tech in the sense that it refers to the rationality of agents, but not to CK of rationality or of the model:[18] the criterion focuses on agents' actions which depend only on their beliefs about the state of the system, and not on their beliefs about other agents' beliefs. To put it another way, the criterion appeals only to the results of the first step of the high-tech criterion and emphasizes the one-step elasticity of realizations to expectations. However, we have argued above that it is intuitively plausible that the high-tech and low-tech criteria turn out to be equivalent in this abstract setting, as previously stressed in the literature.[19]

Finally, some words are in order concerning the connections between the eductive viewpoint and the "evolutive" or adaptive learning viewpoint. At this point let us only say that the failure to find a set $V(E^*)$ for which the equilibrium is locally eductively stable signals a tendency for any near-equilibrium states of beliefs, a priori reachable through some "reasonable" evolutive updating process, to be driven away in some cases, a fact that threatens the convergence of the corresponding learning rule.[20]

[16] The conjectural equilibrium bounds discussed by Benhabib and Bull (1988), in the context of the overlapping generations model of money, has a similar motivation.

[17] Equivalently, in the absence of such a neighborhood $V(E^*)$, facts may falsify any "collective" conjecture, whatever the proximity of the conjectured set to the equilibrium (unless the conjecture is reduced to the equilibrium E^* itself).

[18] It does not even require full knowledge of the model.

[19] A formal statement of the equivalence requires additional technical assumptions such as the (weak) assumptions stressed of Guesnerie and Jara-Moroni (2011). Their results also allow one to show that the local analysis may concentrate on heterogenous point-expectations.

[20] And certainly forbids a strong form of "monotonic" convergence. More on this subject can be found in Guesnerie (2002), Guesnerie and Woodford (1991) and Gauthier and Guesnerie (2004).

3.3 Example: Cobweb Model

To illustrate the eductive approach, consider the cobweb model, interpreted as a producers' game in which the strategy of each firm is its output and the optimal choice of output depends on expected price. This example is analyzed in detail in Guesnerie (1992). For simplicity, consider a static, nonstochastic model and assume that market demand $d(p)$ is decreasing in price p and that for each firm the cost of producing quantity q is $(2C)^{-1}q^2$, with $C > 0$. There is a continuum of firms $\omega \in [0, 1]$. Each firm ω must make its production decision before the price is known, and chooses its quantity $q(\omega)$ based on expected price $p^e(\omega)$. Thus $q(\omega) = Cp^e(\omega)$ and with market clearing $d(p) = \int q(\omega)d\omega = C \int p^e(\omega)d\omega$. There is a unique perfect foresight equilibrium $\bar{p}$, satisfying $d(\bar{p}) = C\bar{p}$, and $p^e(\omega) = \bar{p}$ for all ω. For small deviations in expected price, the equilibrium price p is given to first order by

$$p = \mu - \phi \int p^e(\omega)d\omega,$$

for appropriate μ and where $\phi = -C/d'(\bar{p}) > 0$.[21] We now ask whether individually rational agents endowed with common knowledge, as previously discussed, would necessarily coordinate on rational expectations.

The eductive argument works as follows. Let $V(\bar{p})$ denote a small open interval containing $\bar{p}$. Suppose all agents believe that $p \in V(\bar{p})$. It follows that $p \in \mathcal{E}(1) \equiv \phi V(\bar{p})$, and if $\phi < 1$ then $\mathcal{E}(1)$ is a proper subset of $V(\bar{p})$, so that low-tech eductive stability is implied. Continuing in the stable case, if, in addition, it is hypothetically common knowledge that $p \in V(\bar{p})$, then $p^e(\omega) \in V(\bar{p})$ for all ω and this is also common knowledge. It then follows that it is common knowledge that $p \in \mathcal{E}(1)$. Iterating this argument it follows that $p^e(\omega) \in \mathcal{E}(N) \equiv \phi^N V(\bar{p})$ for all $N = 0, 1, 2, \ldots$, so that high-tech eductive stability is implied. In contrast, eductive stability fails if $\phi > 1$ since, even under the hypothetical common knowledge assumption $p \in V(\bar{p})$, hyper-rational agents are unable to coordinate on $p = \bar{p}$ through mental reasoning.

3.4 Eductive stability within the RBC model

We now return to the RBC model, and in line with the above discussion we begin with the high-tech viewpoint. The time dimension of our problem, and in particular the infinite horizon, as well as the fact that agents are infinitely-lived, brings some additional features to our general framework and then raises some additional issues.

The equilibrium E^* under consideration is given by $K_t = \bar{K}$ for all t, and the first issue is concerned with the notion of a neighborhood $V(E^*)$, which is less straightfor-

[21] The case of positive feedback $-\phi > 0$ arises in some monetary models and can also arise in the "cobweb" setting if, for example, $d'(\bar{p}) > 0$. Bray and Savin (1986) allow for this possibility. Allowing for either sign of ϕ the eductive stability becomes $|\phi| < 1$.

ward here than in timeless or short-horizon contexts. We begin with a simple, natural restriction on the initial time paths of beliefs, that they lie within an ε-neighborhood of the steady state, which might be called the "cylinder" assumption. In later sections we consider alternative initial assumptions. The cylinder assumption is:

B1: For some $\varepsilon > 0$ sufficiently small, $K_t \in D(\varepsilon) \equiv [\bar{K} - \varepsilon, \bar{K} + \varepsilon]$ for all t.[22]

Recall that we restrict attention to paths close enough to the steady state so that first-order approximations are adequate. Thus here we take ε to be small enough so that first-order approximations remain valid. Under the high-tech eductive approach, we assume that the agents are rational, that they know the model (including the connections between r, w and K), that they know the fact that there is a continuum of identical rational agents, and that these facts are common knowledge. Finally, we hypothetically assume that B1 is common knowledge:

CK1: It is hypothetically Common Knowledge that B1 is satisfied.

Given our emphasis on point expectations, which is justified for a small neighborhood of beliefs, a point expectations beliefs profile K^e, as introduced above, satisfies B1 provided that $K_t^e(\omega) \in D(\varepsilon)$ for all ω and t. Furthermore, with CK1, we assume that agent ω' knows that $K_t^e(\omega) \in D(\varepsilon)$, and that agent ω'' knows that agent ω' knows that $K_t^e(\omega) \in D(\varepsilon)$, etc.

Definition 2 *The steady state is CK1 strongly eductively stable if CK1, in addition to the above assumptions, implies that it is common knowledge that the equilibrium path, $K_t = \bar{K}$ for all t, will take place.*[23]

On occasions when it is necessary to be explicit we index B1 by B1(ε) and CK1 by CK1(ε).

We denote by $dK_t = \int dk_t(\omega) d\omega$ the time t aggregate planned savings of agents, where we use our first-order notation for convenience. A proof of strong eductive stability would then proceed as follows. The first stage of the mental process places attention on the possible aggregate plans $dK = \{dK_t\}_{t \geq 0}$ generated by the infinite-horizon plans $\{dk_t(\omega)\}_{t \geq 0, \omega \in [0,1]}$ associated with a profile $dK^e = \{dK_t^e(\omega)\}_{t \geq 0, \omega \in [0,1]}$ of initial beliefs satisfying B1. If these possible paths dK restrict the initial beliefs, then the process continues given these new belief restrictions, and so on. If this mental process converges, the agents' CK beliefs about the aggregate path coincide with the

[22] We will later introduce the alternative assumption B2.

[23] We slightly depart from previous terminology, by leaving the local aspects of the analysis implicit (and not referring to local eductive stability) and by using the word "strong" to stress that the simultaneous coordination at time zero concerns the entire time path of aggregate capital. Later we will distinguish this notion of stability from less demanding concepts discussed below.

equilibrium path: they correctly predict the next period's state and all future states resulting from the guessed actions and plans; in our terminology, the steady state is strongly eductively stable. Hence, in our intertemporal context, while the high-tech story is more sophisticated than it is in the abstract timeless or one-period context introduced above, it follows the same logic: rational agents relying on CK1 are able to deduce at time zero that $K_t = \bar{K}$ for all t.

We now move to a low-tech interpretation for our model consistent with Section 3.2, which is given by the following definition:

Definition 3 *We say that $\bar{K}$ is B1 strongly stable under the low-tech viewpoint if beliefs consistent with B1 imply aggregate plans consistent with B1.*

In the timeless environment examined in Sections 3.1-3.2, aggregate plans are fully determined by the beliefs of agents, making the low-tech implementation unambiguous. In our dynamic setting, however, the realizations of the aggregate over time can differ from agents' expectations, leading them to change their plans, and possibly to revise their expectations of the future. We will come to the question of revised expectations later in Section 5.

At this stage, we introduce an alternative low-tech interpretation in our model, which emphasizes the real-time evolution of the economy when initial beliefs $\{K_t^e(\omega)\}_{t=1}^{\infty}$ are strictly maintained as the economy evolves. We restrict the corresponding definition to the case of point expectations beliefs.

Definition 4 *We say that $\bar{K}$ is B1 strongly stable, under the low-tech viewpoint, in the alternative sense, if beliefs consistent with B1 and maintained over time imply aggregate trajectories consistent with B1, i.e*

$$\{K_s\}_{s=1}^{\infty} \in D^{\infty}(\varepsilon) \equiv D(\varepsilon) \times D(\varepsilon) \times \cdots$$

when each agent ω chooses $c_t(\omega), k_{t+1}(\omega)$ optimally given his savings $k_t(\omega)$, K_t and his expectations $\{K_s^e(\omega)\}_{s=t+1}^{\infty}$.

There are close connections between these three notions of stability. In particular,

Proposition 1 *(i) A failure of strong stability under the first definition of the low-tech viewpoint (Definition 3) implies a failure of strong stability in the CK1 sense (Definition 2).*
(ii) A failure of strong stability under the first definition of the low-tech viewpoint (Definition 3) implies a failure of strong stability in the alternative sense (Definition 4).

Proof. (i) is straightforward. If stability in the sense of Definition 3 does not hold, agents are thus unable to deduce from CK1 that the path for capital must remain in $D(\varepsilon)$. The hypothetical common knowledge assumption cannot be confirmed, the mental process cannot go further and eductive stability fails.

(ii) (sketch) Because there are no income effects associated with changes in aggregate capital, it follows from the Preparation Lemma that the plans of agent ω are to first order independent of aggregate capital. The same is true of the the actual decisions for unchanged beliefs. Hence, if the aggregate plans considered in Definition 3 went outside $D(\varepsilon)$, the same would be true for the actual trajectory.[24] ■

In the following section, we will rely on this proposition to stress that a failure of strong stability under the first low-tech definition implies a failure of strong stability in all senses.

4 The impossibility of Strong (Local) Eductive Stability

From the above analysis, a prerequisite to strong stability, whatever its definition, is the following: under the beliefs restrictions B1, the optimal plans of agents necessarily lead to consumption and saving decisions $c_0(\omega), k_1(\omega)$ that imply $K_1 \in D(\varepsilon)$. This condition, clearly a necessary condition, will be called *weak eductive stability.*

Definition 5 *The steady state $\bar{K}$ is weakly eductively stable if, given the initial condition $k_0(\omega) = \bar{K}$ for all agents ω, and given that all agents' beliefs satisfy B1, then the aggregate capital stock in period $t = 1$ implied by the agents' optimal plans satisfies $K_1 \in D(\varepsilon)$.*

We are now going to focus attention on weak educative stability and then strong educative stability. The principle of our investigation is the following: we focus on individual agents ω noting that given their expectations of the aggregate capital stock $\{dK_t^e(\omega)\}_{t=1}^{\infty}$ they formulate optimal dynamic consumption and savings plans, $\{dc_t(\omega)\}_{t=0}^{\infty}$ and $\{dk_t(\omega)\}_{t=1}^{\infty}$. These individual plans generate an aggregate consumption trajectory

$$\left\{dC_t = \int_0^1 dc_t(\omega)d\omega\right\}_{t=0}^{\infty},$$

and hence a corresponding trajectory for the aggregate capital stock $\{dK_t\}_{t=1}^{\infty}$. We may note that if all agents hold the same expectations then $dC_t = dc_t(\omega)$ and $dK_t = dk_t(\omega)$, and in particular the map Γ^k determines the implied path of aggregate capital.

[24] Indeed, with some additional assumptions the three definitions of strong stabilty are equivalent when ε is small enough. The required additional assumptions for the equivalence would have the same flavor as those used in Guesnerie-Jara-Moroni (2011).

4.1 Weak eductive stability as a necessary condition

To establish our weak stability result we use method just described and note that by Corollary 1 the worst-case expectations are easily identified: they arise when agents believe that the capital stock will remain at one of the boundaries of the cylinder. We have

Theorem 1 *Under B1 the steady state is weakly eductively stable if* $\xi < 1$ *(and only if* $\xi \leq 1$*).*

Proof. From Lemma 4 we have

$$|dK_1| \leq \int |dk_1(\omega)|\, d\omega \leq \xi \sum_{T \geq 1} \left(\beta^{T-1} - \beta^T\right) \int |dK_T^e(\omega)|\, d\omega \leq \xi\varepsilon.$$

This establishes sufficiency. On the other hand, if $\xi > 1$ we can apply Corollary 2 with $dK_T^e(\omega) = \varepsilon$ for all ω and $T = 1, \ldots, N$ and $dK_T^e(\omega) = 0$ for $T \geq N+1$ to get

$$dK_1 = -\xi(1 - \beta^N)\varepsilon.$$

The result follows. ■

We have earlier stressed the key role of the parameter ξ in relating realizations to expectations. The condition $\xi < 1$ implies that high σ and low f'' promote eductive stability. This makes intuitive sense: if f'' is small then a given change dK^e results in a change in the expected interest rate dr^e that is small in magnitude. If σ is large then the substitution effect is small and a given change dr^e leads to a small change $dc_0(\omega)$. Thus when f'' is small and σ is large, a given change dK^e will result in a small change dC_0 in consumption at $t = 0$ and hence a small change in first-period capital dK_1.

As an example, suppose that production is Cobb-Douglas, so that $f(K) = K^\varphi$, for $0 < \varphi < 1$, and that utility takes the constant elasticity form $U(c) = (c^{(1-\sigma)} - 1)/(1-\sigma)$, for $\sigma > 0$. Then

$$\bar{K} = \left(\frac{\varphi}{\bar{r} + \delta}\right)^{1/(1-\varphi)} \text{ and } f'' = \varphi(\varphi - 1)\bar{K}^{\varphi - 2}.$$

Using these and the steady-state equations $\bar{r} = \beta^{-1} - 1$ and $\bar{C} = \bar{K}^\varphi - \delta\bar{K}$, it can be computed that if, say, $\varphi = 1/3$, $\bar{r} = 0.05$ and $\delta = 0.10$, then we have weak eductive stability if and only if $\sigma > 2/3$. This condition is perhaps plausibly satisfied, but we have obtained the somewhat surprising result that even weak eductive stability of the RBC model cannot be taken for granted.

4.2 Strong eductive stability: impossibility theorems

We now turn to our central results in which we establish the failure of local strong eductive stability. For convenience we will often drop the word "local" and refer simply to strong eductive stability.

4.2.1 The impossibility of CK1 strong eductive stability

In Theorem 1 we gave a condition for a minimal consistency requirement: given beliefs B1, the initial plans of agents will necessarily be consistent with B1 *in the first period* $t = 1$ whenever the stated condition $\xi < 1$ is satisfied. However, this is only a weak necessary condition for coordination on the full equilibrium path. Recall that $D(\varepsilon) = [\bar{K} - \varepsilon, \bar{K} + \varepsilon]$. For strong eductive stability of the steady state under B1(ε), it is necessary that the aggregate planned trajectories of aggregate capital lie in $D(\varepsilon)$ – indeed, for the reasons sketched in Section 3.1, it is "almost" sufficient that the following condition be met: for every $\varepsilon > 0$ sufficiently small, under B1(ε) the implied trajectory of aggregate capital $\{K_t\}_{t=1}^{\infty}$ lies in a strictly smaller cylinder $D^{\infty}(\varepsilon')$, i.e. $K_t \in D(\varepsilon')$ for all $t = 1, 2, 3, \ldots$, for some $0 < \varepsilon' < \varepsilon$. Conversely, if for a given beliefs path satisfying B1(ε), the corresponding trajectory of aggregate capital does not remain in $D^{\infty}(\varepsilon)$, this implies eductive instability, as noted above, whichever the definition local eductive instability. In fact, we show

Theorem 2 *Under B1, the steady state is never locally strongly eductively stable, whichever definition is adopted.*

Proof. From Proposition 1, it is enough to prove that strong stability fails in the sense of Definition 3. To this end, we consider the homogeneous belief class in which all agents have the N-period deviation beliefs as given by (11). Setting $0 < e \leq \varepsilon$, it can be seen from (12) that $|dk_N(\omega)| > \varepsilon$ for N sufficiently large. ■

We emphasize the ubiquitous instability implied by Theorem 2 and its generalization given by Theorem 3 below: these results hold for all utility and production functions meeting standard assumptions and for all discount factors $0 < \beta < 1$ and depreciation rates $0 \leq \delta \leq 1$.

We also remark that, for the class of homogeneous N-period deviation beliefs given by (11), for ξ sufficiently small the first N such that $|dk_N(\omega)| > \varepsilon$ can be relatively large. This can be viewed as an extension of our weak eductive stability result: for example, if $e = \varepsilon$, using equation (12), it can be shown that for $t < \xi^{-1}$ aggregate capital dK_t remains inside $D(\varepsilon)$ in accordance with $B1$.[25]

[25] For example, the reader will notice that in the previous Cobb-Douglas economy, $\sigma > 2/3$ was enough to get weak eductive stability. Also, since $\xi = 2/(3\sigma)$ it follows that $\sigma > 3$ implies that the exit time t, just emphasized, is greater than 5.

To summarize the intuition corresponding to Corollary 2, with expectations $dK_s^e = \varepsilon > 0$ for $s \leq N$, the optimal plan for agents is to increase consumption today at the expense of future consumption. For $\xi > 1$ the positive impact on dC_0 is large enough to be immediately destabilizing in the short run in the sense that $dK_1 < -\varepsilon$. However, even for $\xi < 1$, and even if ξ is small, expectations $dK_s^e = e > 0$ for $s \leq N$ for N large imply that *eventually* $dK_t < -\varepsilon$. The planned future reduction in consumption is not enough to avoid, and is consistent with, dK_t leaving $D(\varepsilon)$.

Theorem 2 should be contrasted with the fact that, as noted in the Introduction (and as demonstrated in the Appendix), with short-lived agents eductive stability often holds under natural conditions. The long-run horizon dramatically affects expectational coordination, as evaluated from the strong eductive viewpoint.

In the next Section we show that the instability result does not depend on our specific choice of the neighborhood.

4.2.2 The general impossibility of strong eductive stability

We now consider a more general class of beliefs, which nests B1:

B2: There exists a specified deterministic sequence $\{\varepsilon_t\}_{t=1}^{\infty}$ with $0 < \varepsilon_t < \bar{\varepsilon}$ and $\bar{\varepsilon}$ sufficiently small, such that $K_t \in [\bar{K} - \varepsilon_t, \bar{K} + \varepsilon_t]$ for all t.

Under B2, all of our concepts and definitions of strong eductive stability are modified in the obvious way. For example we refer to CK2 strong eductive stability when CK1 is replaced in Definition 2 by CK2: it is hypothetically common knowledge that B2 is satisfied.

Theorem 3 *Under B2, the steady state is never locally strongly eductively stable, whichever definition is adopted.*

Proof. As above, it is enough to prove that stability fails in the low-tech sense of Definition 3. We proceed as above. Letting $s_m = \sum_{i=1}^{m} \varepsilon_i$, we observe that for any $\eta > 0$, there is some m so that $\eta s_m > \varepsilon_m$. Indeed, if s_m diverges the result is immediate; otherwise, ηs_m is increasing and $\varepsilon_m \to 0$. Next, fix some horizon $N > 0$, and let agent ω have beliefs $dK_N^e(\omega) = \varepsilon_t$ for $1 \leq t \leq N$, and zero otherwise. His savings decision is $dk^N(\omega) = -\xi \Theta^k \cdot dK_N^e(\omega)$. Noting that the entries of Θ^k are all positive, that the first m entries in row m are increasing, and that $\theta^k(m, 1) = 1 - \beta$ for all m, we have that

$$\left| dk_N^N(\omega) \right| = \xi \sum_{i=1}^{N} \theta^k(N, i) \varepsilon_i \geq \xi(1 - \beta) \sum_{i=1}^{N} \varepsilon_i.$$

Choosing $\eta = \xi(1-\beta)$, and letting N be as large as necessary, the proof is completed by the initial observation, and the sufficiency of considering individual behavior.[26,27] ■

The negative result of Theorem 3 means in particular that the hyper-rationalistic viewpoint of strong eductive stability is never conclusive.[28] Our hyper-sophisticated agents cannot convince themselves that the rational expectations equilibrium will necessarily prevail.

5 Combining eductive and evolutive learning

We have found that the steady state $\bar{K}$ is not strongly eductively stable according to the various definitions given above: beliefs of the B1 or B2 type cannot trigger CK of the equilibrium. If common knowledge of full rationality has no predictive power, boundedly rational considerations seem unavoidable and this suggests introducing real-time learning, At the same time, it is known that, for a suitable and natural class of adaptive learning rules, $\bar{K}$ is locally asymptotically stable. At first sight, this suggests a disconnect between between the adaptive and eductive approaches that is much more significant than previously noted in the literature. However, a better way to consider the connection is to adopt the next viewpoint which combines the two approaches, mixing bounded rationality, along the lines of evolutive or adaptive learning, with considerations on the proximity of beliefs to equilibrium, associated with the eductive viewpoint.

5.1 The framework

We again endow agents with expectations about the future path of the aggregate capital stock. These expectations are restricted to belong to a set, which for convenience we will take to be the set $D(\varepsilon)$ described by B1. This set can here be viewed as describing a collective belief that provides bounds on individual beliefs. Following the eductive approach of this paper, agents' decisions are based on an assessment of

[26] We remark that if the common knowledge assumption for a sequence $\{\varepsilon_t\}_{t=1}^{\infty}$ satisfies CK2 except that $\varepsilon_t = 0$ for some proper subset of times t, the contradiction obtains more directly by focusing attention on such times.

[27] While the given proof exploits the functional form of the matrix Θ^k, in the Appendix we show that this fundamental instability result follows from strategic substitutability of the map taking expectations to actions, which the map Γ^k exhibits.

[28] Instability results also appear in the adaptive learning literature. For example, Howitt (1992) and Evans and Honkapohja (2003) show instability for a class of interest-rate rules in monetary models. However, these models can also suffer from indeterminacy, and stability under adaptive learning can be restored with a suitable choice of interest-rate rule. The generic instability results of the current paper are particularly striking since the RBC model is in general well-behaved.

the whole future,[29] but in line with what we called the alternative sense in Definition 4, we look at the system in real time. Our focus remains on whether the realized path for aggregate capital stays within $D(\varepsilon)$. Now, however, in accordance with the evolutive or adaptive learning viewpoint, the expected trajectory at time t is assumed not only to reflect initial beliefs but also to respond to observed actual capital.

More precisely, we specify a set of adaptive learning rules that determine the way initial expectations change along the real-time trajectory of aggregate capital. Then, coming back to the collective belief interpretation of B1, reminiscent of the eductive approach, we ask if the implied path $\{K_t\}_{t=1}^{\infty}$ will necessarily satisfy B1, i.e. if the collective belief which serves as a frame for the individual beliefs is subject to real time falsification. If for some nonempty subset of adaptive learning rules, falsification is impossible, then we say that the steady state is B1-stable under evolutive learning, and if this occurs for all adaptive learning rules within the set of adaptive learning rules under consideration, we say it is robustly B1-stable under evolutive learning.

5.2 The real-time system

In the real-time system we assume that at each time t each agent ω re-solves their dynamic optimization problem. That is, at each time t agent ω chooses $dk_{t+1}(\omega)$ optimally given their savings, their expectations (which are now revised each period) and the aggregate capital stock. Using Lemma 3, the real-time description of savings behavior is given by

$$dk_{t+1}(\omega) = dk_t(\omega) - \left(\frac{1-\beta}{\beta}\right)\xi \sum_{j\geq 1} \beta^j dK^e_{t,t+j}(\omega), \tag{14}$$

where now $dK^e_{t,t+j}(\omega)$ is the point expectation of aggregate capital in period $t+j$ held by agent ω in period t, which, as implied by the notation, we now allow to evolve over time.[30]

It can be seen from equation (14) that the agent's decision $dk_{t+1}(\omega)$ depends on a single sufficient statistic for $\left\{dK^e_{t,t+j}(\omega)\right\}_{j=1}^{\infty}$, given by

$$d\hat{K}^e_t(\omega) = \beta^{-1}(1-\beta)\sum_{j\geq 1} \beta^j dK^e_{t,t+j}(\omega).$$

[29]In the adaptive learning literature, within infinite-horizon models, this approach has been followed, for example, in Sargent (1993, pp. 122-125), Preston (2006), Eusepi and Preston (2008) and Evans, Honkapoha and Mitra (2009). An alternative approach in the adaptive learning literature is based on one-step-ahead "Euler equation" learning. See, e.g. Evans and Honkapohja (2001), Ch. 10, and Evans and McGough (2012).

[30]The proofs of Lemmas 1 - 3 show that the formulae may be applied at any time t, using time t expectations of future aggregate capital, the time t realization of aggregate capital, and the agent's time t savings, provided these quantities are near the steady state $\bar{K}$.

The normalization factor $\beta^{-1}(1-\beta)$ ensures that the sum of the weights on $dK^e_{t,t+j}(\omega)$ is one, so that under B1, $d\hat{K}^e_t(\omega)$ must lie in the interval $[-\varepsilon, \varepsilon]$. We thus rewrite (14) as

$$dk_{t+1}(\omega) = dk_t(\omega) - \xi d\hat{K}^e_t(\omega),$$

and, following the boundedly rational adaptive learning approach, interpret this equation as providing a behavioral decision rule in which we re-envision the agent's saving choice as depending solely on current wealth and the sufficient statistic $d\hat{K}^e_t(\omega)$. For convenience we refer to $d\hat{K}^e_t(\omega)$ as "expected future capital." Finally we specify a simple adaptive scheme for the revisions of $d\hat{K}^e_t(\omega)$ over time:

$$d\hat{K}^e_t(\omega) = (1-\alpha)d\hat{K}^e_{t-1}(\omega) + \alpha dK_t,$$

where $0 < \alpha \leq 1$ parameterizes how expectations adapt to current information about the actual capital stock.[31]

We are now in a position to describe the real-time evolution of the system. For the sake of simplicity, we start from an initial situation in which the time-zero belief is the same for everybody:

$$d\hat{K}^e_0(\omega) \equiv \beta^{-1}(1-\beta) \sum_{j \geq 1} \beta^j dK^e_{0,j}(\omega) = d\hat{K}^e_0,$$

and thus the sufficient statistics are initially, and remain, homogeneous across agents. Finally, in line with our earlier analysis, we assume that $dK_0 = 0$.[32]

The homogeneity assumption allows us to calculate the resulting time path, and is also illuminating in the sense that we would hope the system to be stable under learning if we start with small expected future capital $d\hat{K}^e_0$. Under homogeneity the system can be written as

$$dK_{t+1} = dK_t - \xi d\hat{K}^e_t \tag{15}$$
$$d\hat{K}^e_{t+1} = (1-\alpha)d\hat{K}^e_t + \alpha dK_{t+1}. \tag{16}$$

We can now return to the previously suggested definitions of B1-stability under adaptive learning and give formal definitions.

[31] Adaptive learning in nonstochastic models with infinite horizons often assumes "steady state learning" in which forecasts are the same at all horizons. See, for example, Evans, Honkapohja and Mitra (2009). In the current context this would mean $dK^e_{t,t+j} = e$ at t for all j, with the value of e updated over time. Our formulation in terms of $d\hat{K}^e_t$ allows for greater generality, while retaining a single sufficient statistic that is updated over time. In stochastic models, the time pattern of interest rates can be estimated and updated using recursive least squares. For technical reasons this procedure cannot be used in nonstochastic systems. Intuitively, in a nonstochastic equilibrium the asymptotic lack of temporal variation makes impossible consistent estimation of the time-series parameters. See Evans and Honkapohja (2001, pp. 152-154).

[32] Our results are qualitatively robust to small perturbations of intial aggregate capital.

Definition 6 *The equilibrium is B1-stable under adaptive learning for a given* $0 < \alpha \leq 1$ *if, for all* $\varepsilon > 0$, $|d\hat{K}_0^e| < \varepsilon$ *implies that the trajectory* $\{dK_t\}_{t=1}^{\infty}$, *generated by (15)-(16), remains in the cylinder* $D(\varepsilon)^{\infty}$.

Definition 7 *The equilibrium is robustly B1-stable under adaptive learning if it is B1-stable under adaptive learning for all* $0 < \alpha \leq 1$.

If an equilibrium is B1-stable for some nonempty subset of $0 < \alpha \leq 1$, but is not robustly B1-stable, then we will say that it is partially B1-stable under adaptive learning.

5.3 The results

The first result is again an impossibility theorem.

Theorem 4 *The equilibrium cannot be robustly B1-stable under adaptive learning.*

The proof is given in the Appendix. In fact the result should be expected in view of Theorem 2. Consider the optimal plans from Section 2.2 generated by the beliefs $dK_t^e(\omega) = d\hat{K}_0^e \neq 0$ for $1 \leq t \leq N$, and $dK_t^e(\omega) = 0$ otherwise, for large N. Over any given finite stretch of time τ, for small $\alpha > 0$ the trajectory under adaptive learning will be close to these plans. But from Theorem 2 we know that this trajectory of plans exits the cylinder $D(\varepsilon)^{\infty}$.

A striking feature of this result is that instability arises for small α, which in the adaptive and least-squares learning literature is usually viewed as a stabilizing case (the "small gain" limit).[33] In our approach, the problem is that in this case the initial collective belief will be falsified, which we view as a fragility of expectational coordination.

The instability result is stronger than stated in the following sense, which can again be verified from the proof of Theorem 4. Given $\varepsilon > 0$, consider any initial beliefs $0 < |d\hat{K}_0^e| < \varepsilon$. Then there exists $\alpha > 0$ such that the corresponding trajectory under adaptive learning will have $|dK_t| > \varepsilon$ for some $t \geq 1$. That is, starting from the

[33] The connection between eductive stability and stability under evolutive (or adaptive) learning rules, has been discussed, for example, in Evans and Guesnerie (1993), Guesnerie (2002) and Hommes and Wagener (2010). In short-horizon set-ups, eductive instability is usually reflected in adaptive instability for large gains (here $\alpha < 1$ large). This is seen for the overlapping generations model with money, under adaptive learning, in Guesnerie and Woodford (1991) and Evans and Honkapohja (1995), and for the cobweb model under dynamic predictor-selection learning, this connection is apparent in Brock and Hommes (1997). Theorem 4 is thus particularly unexpected in that it establishes strong instability under adapative learning for small $\alpha > 0$.

steady state, all initial expectations, for appropriate α, lead to paths dK_t that leave $D(\varepsilon)$.

We next turn to partial B1-stability under adaptive learning. We will see that a natural necessary condition is given by asymptotic stability of the system (15)-(16). Asymptotic stability is the classical stability criterion used for adaptive learning. We therefore start with the following result:

Lemma 5 *The evolutive system (15)-(16) is asymptotically stable if and only if*

$$\xi < 4\alpha^{-1} - 2.$$

Lemma 5 implies asymptotic stability for all $0 < \alpha < 1$ if $\xi < 2$ and for some $0 < \alpha < 1$ if $\xi \geq 2$. For given $\varepsilon > 0$, asymptotic stability implies that $|dK_t|, |d\hat{K}_t^e| \leq \varepsilon$ for t sufficiently large. In fact asymptotic stability is a necessary condition for partial B1-stability under adaptive learning.[34] However this is only one necessary condition for partial B1-stability under adaptive learning. It is immediate from (15) that $\xi < 1$ is another necessary condition.

We can summarize the results in the following Theorem.

Theorem 5 *Under adaptive learning the partial B1-stability and the asymptotic stability properties of (15)-(16) are as follows:*

1. *For* $0 < \xi < 1$, *the steady state is asymptotically stable for all* $0 < \alpha < 1$ *but is only partially B1-stable under adaptive learning. Specifically, given* $0 < \xi < 1$, *for sufficiently small* α *the the steady state is not B1-stable, while for sufficiently large* α *the the steady state is B1-stable.*

2. *For* $1 < \xi$, *the steady state is not partially B1-stable under adaptive learning, although:*

   (a) *For* $1 \leq \xi < 2$, *the learning process is asymptotically stable whatever* α.

   (b) *For* $2 \leq \xi$, *the learning process is asymptotically stable for* $\alpha < \frac{4}{2+\xi}$.

This theorem emphasizes the relevance of the expectations feedback parameter for the understanding of real-time learning. Indeed, the coefficient ξ, stressed in Theorem 1, plays a key role both in the understanding of B1-stability under adaptive learning

[34] To see this, assume the system is not asymptotically stable. It can be shown that the eigenvalues in this case are real. (The proof of Lemma 5 implies that in the complex case the modulus of the eigenvalues is $1 - \alpha$.) We must show that $dK_0 = 0$, $d\hat{K}^e \neq 0$ leads to a divergent path. Noting that $(0, 1)$ is not an eigenvector it follows that all initial conditions $(0, d\hat{K}^e)$ lead to divergent paths provided at least one of the eigenvalues has magnitude larger than one.

and in the analysis of the asymptotic stability of the adaptive learning process. When $\xi > 2$ asymptotic stability obtains only for $\alpha > 0$ sufficiently small, while for $\xi < 2$ asymptotic stability holds for all $0 < \alpha < 1$. If $\xi > 1$ then partial B1-stability fails: for some initial beliefs satisfying B1, the path will falsify these beliefs, *whatever* the specific (asymptotically stable or not) adaptive learning rule used by the agents. In contrast if $\xi < 1$ then B1-stability holds for α sufficiently large.[35]

Finally, we emphasize that while for given ξ the asymptotic stability condition $\xi < 4\alpha^{-1} - 2$ is easier to satisfy when $\alpha > 0$ is small, it is small values of α that generate the failure of robust B1-stability under adaptive learning. Small $\alpha > 0$ under adaptive learning leads to a cumulative movement of aggregate capital away from the steady state value, which over finite time periods, as $\alpha \to 0$, track the possible K_t paths deduced by agents in our eductive setting.[36]

6 Discussion

In Section 4 we showed a generic failure of strong eductive stability in the RBC model. The implication is that full coordination on the REE, as of time zero, cannot be expected, based on full rationality and on common knowledge of the model and of the rationality of other agents. The conclusion we draw is that coordination on RE is necessarily fragile, and thus deviations from REE can be expected.

The strong eductive instability results of this paper arise from the long planning horizon of agents in the RBC model. This is reflected in our proofs, which show that the violation of the CK assumption obtains not because dK_1 is too large, but because, eventually, dK_t is too large. However, one might still ask whether strong instability would arise in a model with capital in which agents are short-lived but the economy is infinitely-lived. In the Appendix we consider this question by conducting eductive stability analysis in an overlapping generations models. This model can also be interpreted as a variant of our model in which agents are myopic (i.e. they only consider next period as the relevant horizon) instead of being far-sighted and envisaging the whole future. We find that myopia makes expectational coordination easier, and in particular we obtain a strong eductive stability condition that is satisfied for one frequently employed parametric class of models.

[35] Partial B1-stability results for $\xi < 1$ exhibit a trade-off between ξ and α: as $\xi \to 1$ from below, the partial B1-stability region tends to $\alpha > 1/2$. Numerical results indicate partial B1-stability for $\alpha \in (\gamma(\xi), 1]$ where $\gamma(\xi)$ is continuous and monotonically increasing in ξ with $\gamma(\xi) \to 0$ as $\xi \to 0$. When $\xi = 1$ we have partial B1-stability for $\alpha \in [1/2, 1]$. We note a discontinuity in B1-stability for given α, which is caused by first-period behavior: $dK_1 = -\xi dK_0^e$ implies B1-instability for any $\xi > 1$.

[36] An interesting feature of the lack of robust B1-stability under adaptive learning, which can be seen in the proof of Theorem 4 given in the Appendix, is that the instability is associated with long cyclical movements in K_t.

Returning to our RBC framework, a logical consequence of the failure of strong eductive stability is that the evolution of the economy is likely to incorporate adaptive learning rules. In Section 5 we therefore considered paths generated by adaptive learning, and asked whether their dynamics reflect the eductive instability result. We find that they do. We have given a number of adaptive learning results that link real-time adaptive learning dynamics to the expectations feedback parameter ξ, and also, most strikingly, we have provided a strong adaptive learning result that is closely linked to the strong eductive instability: Theorem 4 shows that while real-time adaptive learning under small gain is asymptotically stable, the resulting time paths will include periods of large deviations from RE, even if initial expectations are close to the steady state. This result suggests that non-negligible deviations from the REE are likely. Starting from a steady state, suppose there is a small shock to expectations, e.g. created by some news event. Then not only will coordination on the REE be impossible using eductive learning, but also some plausible adaptive learning rules, even if they return the economy to the steady state asymptotically, will necessarily first lead the economy *away* from the steady state.

Our results also show the key role of the expectations feedback parameter ξ for both eductive and adaptive considerations. Section 4 began by establishing weak eductive stability as necessary for expectational coordination, and found that $\xi < 1$ was the required condition. The parameter ξ is additionally central to an extended notion of weak eductive stability: see the remarks following Theorem 2. It follows from the discussion there that, for fixed N and given homogeneous N-period deviation beliefs (11), the implied path of aggregate plans stays inside the ε-cylinder when the expectational feedback ξ is sufficiently small. Clearly ξ also plays a key role in the evolutive learning results of Section 5, as asymptotic stability is implied when $\xi < 2$. It can further be shown from Theorem 4 that, for given gain α, the implied path of aggregate capital stays inside the ε-cylinder when the expectational feedback ξ is sufficiently small.[37] Thus, for fixed α, small ξ implies B1-stability.

Returning to our findings of likely non-negligible deviations from REE, we note that the adaptive learning literature has been moving beyond a sole focus on the stability of REE and now includes investigations of possible additional learning dynamics: see the numerous illustrations provided in Evans and Honkapohja (2009). An example is the dynamic predictor selection approach of Brock and Hommes (1997), in which they show that exotic learning dynamics around the REE can arise in the cobweb model. As noted in our discussion in Section 3.3, the REE in the cobweb model is not eductively stable when the expectations feedback parameter ϕ satisfies

[37] For ξ sufficiently small, the eigenvalues of the system (15)-(16) are real and positive. The proof of Theorem 4 shows that, in this case, the path dK_t is proportional to $(\lambda_1 - \lambda_2)(\lambda_1^t - \lambda_2^t)$, which, as a continuously differentiable function of t, has at most one critical point for $t > 0$. This, together with the fact that the maximum magnitude of dK_t is continuous in ξ, and approaches zero as $\xi \to 0$, yields the result.

$\phi > 1$. The parameter values for which complex real-time learning dynamics arise in Brock and Hommes (1997) correspond precisely to this case. See also Hommes (2011) for related experimental evidence.[38] Thus the eductive viewpoint may be viewed as opening another door to the new territories explored in learning studies, providing a powerful tool for their exploration.

7 Conclusions

The difficulties of expectational coordination can be ascertained from two sides, the eductive one and the evolutive or adaptive one. In both cases, farsighted agents are sensitive to the whole path of expectations and long-run expectations significantly matter. It is not surprising that long-run concerns influence present decisions and future plans. But the sensitivity to expectations of long-run plans envisaged today is extreme, and this is at the heart of the impossibility of strong eductive stability. Indeed, in the eductive framework there is negative short-run feedback, which *may* be destabilizing, while in the long run strategic substitutability is decisive and the negative feedback of expected capital *will be* destabilizing. That is, given an expectation that aggregate K will fall below $\bar{K}$ by a small amount for a long enough stretch of time, the optimal dynamic plans of agents call for them to eventually accumulate capital in excess of $\bar{K}$ by a larger amount. Hypothetical Common Knowledge of the equilibrium cannot trigger Common Knowledge of the equilibrium, whatever the specific characteristics of the economy, an extreme form of expectational instability that has no counterpart in previously studied models.[39] If evolutive learning is incorporated into the model, so that expectations evolve adaptively over time, these sources of instability remain pivotal. If the adaptation parameter is large then unstable overshooting can arise in the short run, while if the adaptation rate is small then low-frequency swings over the medium run will necessarily generate instability.

[38] For RBC models, potentially important adaptive learning dynamics have been noted in Evans, Honkapohja and Mitra (2009), Eusepi and Preston (2011) and Branch and McGough (2011).

[39] We do not claim however that the difficulty occurs in every model with infinitely lived agents. For example, it would not occur in the world of Lucas (1978), where the assets returns do not depend upon "extrinsic" uncertainty, but only upon intrinsic uncertainty.

Appendix 1

Proof of Lemma 1. The agent's lifetime budget constraint, dated here at time t and with his transversality condition incorporated, is given by

$$\begin{aligned} \sum_{n=0}^{\infty} R_t^n c_{t+n}(\omega) &= \sum_{n=0}^{\infty} R_t^n q_{t+n} + (1+r_t)k_t(\omega), \text{ where} \\ R_t^n &= \prod_{i=1}^{n} (1+r_{t+i})^{-1},\ R_t^0 = 1. \end{aligned} \tag{17}$$

We now compute the total derivative of (17) at the steady state. Noting from $dR_t^n = -\beta^{n+1} \sum_{i=1}^{n} dr_{t+i}$ that

$$\sum_{n \geq 0} dR_t^n = -\frac{1}{\bar{r}} \sum_{n \geq 1} \beta^n dr_{t+n},$$

we obtain

$$\sum_{n \geq 0} \beta^n dc_{t+n}(\omega) = \beta^{-1} dk_t(\omega) + dq_t + \bar{K} dr_t + \sum_{n \geq 1} \beta^n dq_{t+n} + \left(\frac{\bar{C} - \bar{q}}{\bar{r}} \right) \sum_{n \geq 1} \beta^n dr_{t+n}. \tag{18}$$

In steady state, we have $\bar{C} = \bar{q} + \bar{r}\bar{K}$. Also, by constant returns to scale, $q = f(K) - Kf'(K)$, so that $dq + \bar{K}dr = 0$. It follows from equation (18) that

$$\sum_{n \geq 0} \beta^n dc_{t+n}(\omega) = \beta^{-1} dk_t(\omega) + \sum_{n \geq 1} \beta^n (dq_{t+n} + \bar{K} dr_{t+n}) = \beta^{-1} dk_t(\omega).$$

Setting $t = 0$ gives the result claimed in Lemma 1.[40] ∎

Proof of Lemma 2. From the agent's flow budget constraint (2) we have, to first order, that

$$dk_{t+1}(\omega) = (1+\bar{r}) dk_t(\omega) + \bar{K} dr_t + dq_t - dc_t(\omega)$$

from which (7) follows using the zero income effect $dq + \bar{K}dr = 0$. Under point expectations equation (3) is $U'(c_t(\omega)) = \beta(1 + r_{t+1}^e(\omega)) U'(c_{t+1}(\omega))$. Thus to first order

$$U''(\bar{C}) dc_t(\omega) = \beta(1+\bar{r}) U''(\bar{C}) dc_{t+1}(\omega) + \beta U'(\bar{C}) dr_{t+1}^e(\omega).$$

Using $dr_{t+1}^e(\omega) = f''(\bar{K}) dK_{t+1}^e(\omega)$ together with $\beta(1+\bar{r}) = 1$ and the definition of σ gives (8). ∎

Proof of Lemma 3. By recursive substitution the linearized Euler equation (8) taken at time t gives

$$dc_{t+n}(\omega) = dc_t(\omega) + \frac{\beta \bar{C} f''}{\sigma} \sum_{s=1}^{n} dK_{t+s}^e(\omega).$$

[40] In fact, the Lemma holds even if production exhibits decreasing returns, provided that factor markets are competitive. Also, an alternative proof is available based on the envelope theorem.

Substituting into the Welfare Corollary 1 taken at time t, i.e.

$$\beta^{-1} dk_t(\omega) = \sum_{n \geq 0} \beta^n dc_{t+n}(\omega),$$

gives

$$\begin{aligned} \beta^{-1} dk_t(\omega) &= (1-\beta)^{-1} dc_t(\omega) + \frac{\beta \bar{C} f''}{\sigma} \sum_{n \geq 1} \beta^n \sum_{s=1}^{n} dK^e_{t+s}(\omega) \\ &= (1-\beta)^{-1} dc_t(\omega) + \frac{\beta \bar{C} f''}{(1-\beta)\sigma} \sum_{n \geq 1} \beta^n dK^e_{t+n}(\omega), \end{aligned}$$

which yields the optimal plan for consumption

$$dc_t(\omega) = \left(\frac{1-\beta}{\beta}\right) dk_t(\omega) - \frac{\beta \bar{C} f''}{\sigma} \sum_{n \geq 1} \beta^n dK^e_{t+n}(\omega),$$

which is equivalent to (9). Combining this with the linearized flow budget constraint, $dk_{t+1}(\omega) = \frac{1}{\beta} dk_t(\omega) - dc_t(\omega)$, yields

$$dk_{t+1}(\omega) = dk_t(\omega) + \frac{\beta \bar{C} f''}{\sigma} \sum_{n \geq 1} \beta^n dK^e_{t+n}(\omega), \tag{19}$$

which is equivalent to (10). ■

Proof of Lemma 4. We start with the savings plan**.** It suffices to show that

$$dk_t(\omega) = -\xi \sum_{n \geq 1} \theta^k(t,n) dK^e_n(\omega) \tag{20}$$

satisfies (19), where we recall that

$$\xi = -\frac{\beta^2 \bar{C}}{\sigma(1-\beta)} f'' \ \text{ and } \ \theta^k(t,n) = \begin{cases} 1-\beta^n & \text{if} \quad n \leq t \\ \beta^{n-t}(1-\beta^t) & \text{if} \quad n > t \end{cases}.$$

We compute that $dk_t(\omega) + \frac{\beta \bar{C} f''}{\sigma} \sum_{n\geq 1} \beta^n dK^e_{t+n}(\omega)$

$$
\begin{aligned}
&= -\xi \sum_{n\geq 1} \theta^k(t,n) dK^e_n(\omega) - \xi \left(\frac{1-\beta}{\beta}\right) \sum_{m\geq 1} \beta^m dK^e_{t+m}(\omega) \\
&= -\xi \sum_{n=1}^{t} \theta^k(t,n) dK^e_n(\omega) - \xi \sum_{m\geq 1} \left(\beta^{m-1}(1-\beta) + \beta^m(1-\beta^t)\right) dK^e_{t+m}(\omega) \\
&= -\xi \sum_{n=1}^{t} \theta^k(t,n) dK^e_n(\omega) - \xi \sum_{m\geq 1} \left(\beta^{m-1} - \beta^{t+m}\right) dK^e_{t+m}(\omega) \\
&= -\xi \sum_{n=1}^{t} \theta^k(t+1,n) dK^e_n(\omega) - \xi \sum_{m\geq 1} \beta^{m-1} \left(1 - \beta^{t+1}\right) dK^e_{t+m}(\omega) \\
&= -\xi \sum_{n=1}^{t} \theta^k(t+1,n) dK^e_n(\omega) - \xi \sum_{m\geq 1} \theta(t+1,t+m) dK^e_{t+m}(\omega) \\
&= -\xi \sum_{n\geq 1} \theta^k(t+1,n) dK^e_n(\omega) = dk_{t+1}(\omega).
\end{aligned}
$$

This establishes the first part of Lemma 4.

To prove the second part concerning the consumption plan, we combine (9) from Lemma 3 with (20) to obtain

$$
\begin{aligned}
dc_0(\omega) &= \left(\frac{1-\beta}{\beta}\right) \xi \sum_{n\geq 1} \beta^n dK^e_n(\omega) \\
dc_t(\omega) &= -\left(\frac{1-\beta}{\beta}\right) \xi \left(\sum_{n\geq 1} \theta^k(t,n) dK^e_n(\omega) - \sum_{n\geq 1} \beta^n dK^e_{t+n}(\omega)\right), \text{ for } t \geq 1.
\end{aligned}
$$

It follows that

$$
\theta^c(t,T) = \begin{cases} \theta^k(t,T) & \text{if} \quad T \leq t \\ \theta^k(t,T) - \beta^{T-t} & \text{if} \quad T > t \end{cases},
$$

which completes the proof. ■

Proof of Corollary 2. We apply Lemma 4. For $1 \leq t < N$ we have $dk_t(\omega)$ equals

$-\xi e$ times

$$\begin{aligned}
\sum_{T=1}^{t}\theta^k(t,T)+\sum_{T=t+1}^{N}\theta^k(t,T) &= \sum_{T=1}^{t}(1-\beta^T)+\sum_{T=t+1}^{N}(\beta^{T-t}-\beta^T) \\
&= \sum_{T=1}^{t}(1-\beta^T)+(1-\beta^t)\sum_{T=1}^{N-t}\beta^T \\
&= t-\beta\frac{1-\beta^t}{1-\beta}+(1-\beta^t)\frac{\beta}{1-\beta}(1-\beta^{N-t}) \\
&= t-\bar{r}^{-1}(1-\beta^t)\beta^{N-t}.
\end{aligned}$$

Similarly for $0\le t<N$ we have $dc_t(\omega)$ equals $-\xi e$ times

$$\begin{aligned}
\bar{r}\sum_{T=1}^{t}\theta^c(t,T)+\bar{r}\sum_{T=t+1}^{N}\theta^c(t,T) &= \bar{r}\sum_{T=1}^{t}(1-\beta^T)-\bar{r}\sum_{T=t+1}^{N}\beta^T \\
&= \bar{r}\left(t-\beta\frac{1-\beta^t}{1-\beta}\right)-\bar{r}\beta^{t+1}\frac{1-\beta^{N-t}}{1-\beta} \\
&= \bar{r}t-1+\beta^N.
\end{aligned}$$

For $t\ge N$ we have $dk_t(\omega)$ equals $-\xi e$ times

$$\sum_{T=1}^{N}\theta^k(t,T)=\sum_{T=1}^{N}(1-\beta^T)=N-\frac{\beta}{1-\beta}(1-\beta^N)=N-\bar{r}^{-1}(1-\beta^N)$$

and $dc_t(\omega)$ equals $-\xi e$ times

$$\begin{aligned}
\bar{r}\sum_{T=1}^{N}\theta^c(t,T) &= \bar{r}\sum_{T=1}^{N}(1-\beta^T) \\
&= \bar{r}\left(N-\frac{\beta}{1-\beta}(1-\beta^N)\right) \\
&= \bar{r}\left(N-\bar{r}^{-1}(1-\beta^N)\right),
\end{aligned}$$

which implies $dc_t(\omega)=\bar{r}dk_t(\omega)$. ■

Strategic substitutability and an alternate proof of Theorem 3. As usual, let $dK^e(\omega)=\{dK^e_t(\omega)\}$ be a beliefs path of a given agent. His corresponding savings plan is then given by $dk(\omega)=\{dk_t(\omega)\}=\Gamma^k(dK^e(\omega))$. As noted before, the map Γ^k can be represented as a semi-infinite matrix $\{\Gamma^k_{tm}\}_{t,m\ge1}$. Using this notation, we may write

$$dk_t(\omega)=\sum_{m\ge1}\Gamma^k_{tm}dK^e_m(\omega). \tag{21}$$

The map Γ^k exhibits local strategic substitutability if $\Gamma^k_{tm} < 0$ for all $t, m \in \mathbb{N}$.

Theorem A1 *If* Γ^k *exhibits local strategic substitutability, and if*

$$\eta \equiv \inf_{n>0} |\Gamma^k_{nn}| > 0,$$

then the perfect foresight steady state is never CK2 strongly eductively stable.

Proof. For notational simplicity, we forgo identifying the dependence of beliefs and actions on ω. Let $\varepsilon = \{\varepsilon_n\}$ be any uniformly bounded sequence of non-negative real numbers, with at least one strictly positive entry. Define sequences of expectations $dK^e(n)$ as follows:

$$dK^e_t(n) = \begin{cases} \varepsilon_t & \text{if} \quad t \leq n \\ 0 & \text{if} \quad t > n \end{cases},$$

and let $dk(n) = \Gamma^k\left(dK^e(n)\right)$.

By strategic substitutability, $dk_t(\varepsilon) < dk_t(n) < 0$; thus it suffices to find some n and t so that $|dk_t(n)| > \varepsilon_t$. Next, notice that since $dK^e_{n+s}(n) = 0$ for $s > 0$, it follows that $dk_{n+s}(n) = dk_n(n)$. This allows us to compute

$$\begin{aligned} dk_{n+1}(n+1) &= dk_{n+1}(n) + \Gamma^k_{n+1,n+1} \cdot \varepsilon_{n+1} \\ &= dk_n(n) + \Gamma^k_{n+1,n+1} \cdot \varepsilon_{n+1} \\ &\leq dk_n(n) - \eta \cdot \varepsilon_{n+1}. \end{aligned}$$

Thus, substituting in recursively and using $dk_0 = 0$, we have

$$dk_{n+1}(n+1) \leq -\eta \sum_{m=1}^{n+1} \varepsilon_m.$$

The proof is completed by applying the initial observation in the proof of Theorem 3. ■

We note that by Corollary 1, Γ^k exhibits local strategic substitutability. Since $\inf_{n>0} |\Gamma^k_{nn}| = \xi(1-\beta) > 0$, Theorem A1 yields Theorem 3 as a corollary. Also, Corollary 1 and Theorem A1 focus on the implications of local strategic substitutability precisely because our map Γ^k exhibits this characteristic; however, analogous results with analogous proofs hold for maps that exhibit local strategic complementarity.

Proof of Theorem 4. To examine B1-stability we consider the system (15)-(16) with the assumed initial conditions $dK_0 = 0$ and $dK^e_0(\omega) = e$, where $e = \pm\varepsilon$ for all ω. The dynamics of dK_t can equivalently be written as

$$dK_{t+2} = (2 - \alpha(1+\xi))dK_{t+1} - (1-\alpha)dK_t$$

with $dK_0 = 0$ and $dK_1 = -\xi e$. The eigenvalues of the dynamic system are

$$\lambda_1, \lambda_2 = \frac{1}{2}\left\{2 - \alpha(1+\xi) \pm \sqrt{\alpha}\sqrt{\alpha(1+\xi)^2 - 4\xi}\right\}.$$

If $\alpha(1+\xi)^2 < 4\xi$ these eigenvalues are complex conjugates, and the solution is given by

$$dK_t = -\frac{\xi e}{r \sin \psi} r^t \sin(\psi t),$$

where

$$r^2 = 1 - \alpha \text{ and}$$

$$\psi = \sin^{-1}\sqrt{\frac{4\xi\alpha - \alpha^2(1+\xi)^2}{4(1-\alpha)}}.$$

If $\alpha(1+\xi)^2 > 4\xi$ the eigenvalues are real and distinct[41], and the solution is given by

$$dK_t = -(\lambda_1 - \lambda_2)^{-1}\xi e(\lambda_1^t - \lambda_2^t).$$

To show lack of B1-stability for small $\alpha > 0$, note that for sufficiently small α the roots are complex and at $t = T = \frac{\pi}{2\psi}$

$$dK_T = -\frac{2\xi e}{\sqrt{\alpha}\sqrt{4\xi - \alpha(1+\xi)^2}}(1-\alpha)^{\pi/(4\psi)}.$$

Taking the limit as $\alpha > 0$ tends to zero it can be verified that

$$\lim_{\alpha \to 0} dK_{T(\psi(\alpha))} = \pm\infty$$

where the sign is opposite to the sign of e. It follows that for $\alpha > 0$ sufficiently small we have $|dK_t| > \varepsilon$ for values of t near T. ■

Proof of Lemma 5. The system can be written as

$$\begin{pmatrix} dK_{t+1} \\ d\hat{K}^e_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & -\xi \\ \alpha & 1 - \alpha(1+\xi) \end{pmatrix} \begin{pmatrix} dK_t \\ d\hat{K}^e_t \end{pmatrix}.$$

Let A denote the 2×2 matrix that governs the dynamics. For asymptotic stability we need both eigenvalues within the unit circle. Equivalently (see LaSalle (1986), p. 28) we require $|\det(A)| < 1$ and $|\mathrm{tr}(A)| < 1 + \det(A)$. Since $\det(A) = 1 - \alpha$ the first condition is satisfied for all $0 < \alpha < 1$. Using $\mathrm{tr}(A) = 2 - \alpha(1+\xi)$ leads to the stated condition. ■

[41] We omit the non-generic case of repeated roots.

Proof of Theorem 5. 1. Asymptotic stability is immediate from Lemma 5**.** Failure of B1-stability for small $\alpha > 0$ follows from the proof of Theorem 4. To show B1-stability for large $\alpha < 1$, note that for $\alpha = 1$ the solution is $dK_t = -(1-\xi)^t \xi e$. By continuity $\{dK_t\}_{t=0}^{\infty} \in D(\varepsilon)^{\infty}$ for all $\alpha < 1$ sufficiently large.

2. Lack of partial B1-stability is immediate since $dK_1 = -\xi e$ for $dK_0 = 0$ and $dK_0^e(\omega) = e$, where $e = \pm\varepsilon$ for all ω. The asymptotic stability results follow from Lemma 5. ■

Appendix 2: A finite-horizon model with capital

Consider a two-period OLG model with capital, along the lines of Diamond (1965). Population is constant and normalized to one, and all markets are competitive. Let ω_t be an agent born at time t. He is endowed with one unit of labor, which he supplies inelastically for real wage q_t. He then allocates his income between savings $s(\omega_t) = k(\omega_t)$ and consumption $c_1(\omega_t)$. In period $t+1$, this agent is now old: he rents his savings for net real return r_{t+1}, consumes the gross return, plus his share of profits from ownership of firms, and dies. Thus agent ω_t solves the following problem

$$
\begin{aligned}
& \max E(\omega_t)\left\{u\left(c_1(\omega_t), c_2(\omega_t)\right)\right\} \\
\text{s.t.} \quad & c_1(\omega_t) + s(\omega_t) = q_t && (22) \\
& c_2(\omega_t) = (1 + r_{t+1}^e(\omega_t))s(\omega_t) + \pi(\omega_t) && (23)
\end{aligned}
$$

Notice that when agent ω_t makes his savings decision, he does not know the value of r_{t+1}. Below we assume constant returns to scale production so that $\pi = 0$.

The agent ω_t's first-order condition is given by

$$u_{c_1}\left(c_1(\omega_t), c_2(\omega_t)\right) = \beta(1 + r_{t+1}^e(\omega_t))u_{c_2}\left(c_1(\omega_t), c_2(\omega_t)\right). \tag{24}$$

Equations (22)–(24) may be used to compute the savings decision of agent ω_t based on current and expected future factor prices:

$$s(\omega_t) = s(q_t, r_{t+1}^e(\omega_t)).$$

Firms hire workers and rent capital in competitive factor markets, and employ constant returns to scale technology to manufacture goods: $Y = f(K, L)$; thus profits are zero and factors prices are given by the respective marginal products. Capital is inelastically supplied "in the morning" by the old and depreciation is zero: the capital accumulation equation is given accordingly by

$$K_{t+1} = \int s(\omega_t)d\omega_t = \int s(q_t, r_{t+1}^e(\omega_t))d\omega_t.$$

Assuming agents know the relationship between real interest rates and marginal products, and so form expectations of aggregate capital instead of real interest rates, we have

$$K_{t+1} = \int s\left(f_L(K_t, 1), f_K(K_{t+1}^e(\omega_t), 1)\right) d\omega_t, \tag{25}$$

where $K_{t+1}^e(\omega_t)$ is agent ω_t's forecast of aggregate capital tomorrow. Equation (25) captures the dynamics of the economy: given aggregate capital today and forecasts of aggregate capital tomorrow, the actual value of aggregate capital tomorrow can be determined. It also highlights a key difference between the OLG model and the RBC model: in the OLG model aggregate capital depends only on one-period-ahead forecasts; in the RBC model, aggregate capital depends on forecasts at all horizons.

Common Knowledge and Eductive Stability

To motivate the definition of eductive stability, we consider the following thought experiment at time $t = 0$: Let $\bar{K}$ be a steady state of (25): $\bar{K} = s\left(f_L(\bar{K}, 1), f_K(\bar{K}, 1)\right)$. Assume that at time $t = 0$ every old household has capital $k_0(\omega) = \bar{K}$. This determines the wage, and therefore the income, of the young, as given by $\bar{q}$. Each young agent ω_t forecasts capital stock tomorrow, $K^e(\omega_t)$, and determines his savings decision $s(\bar{q}, f_K(K^e(\omega_t), 1))$. He then contemplates the savings decisions of other agents. We again make the common knowledge CK1:

CK1: It is common knowledge that for some $\varepsilon > 0$ sufficiently small, $K_t \in D(\varepsilon) \equiv [\bar{K} - \varepsilon, \bar{K} + \varepsilon]$ for all t.

These CK beliefs are assumed held by all agents at all times, i.e. for all ω_t for all t.

The definitions of weak and strong eductive stability under CK1 are identical to the definitions given in Sections 3 and 4. We have the following results:

Theorem 6 *The steady state $\bar{K}$ is strongly eductively stable if and only if*

$$\left|\partial \mathbf{s}/\partial \mathbf{r}\left(f_L(\bar{K}, 1), f_K(\bar{K}, 1)\right) \cdot f_{KK}\left(\bar{K}, 1\right)\right| < 1. \tag{26}$$

Proof. To see this, first note that (26) holds if and only if there is $\zeta \in (0, 1)$ such that for small $\varepsilon > 0$, whenever $|K^e(\omega_t) - \bar{K}| \le \varepsilon$ it follows that

$$\left|s\left(f_L(\bar{K}, 1), f_K(K^e(\omega_t), 1)\right) - \bar{K}\right| < \zeta\varepsilon.$$

Under CK1, if (26) holds then $s(\hat{\omega}_0) \in D(\zeta\varepsilon)$ for all $\hat{\omega}_0$. Because each agent ω_0 knows that $K^e(\hat{\omega}_0) \in D(\varepsilon)$, he concludes that $s(\hat{\omega}_0) \in D(\zeta\varepsilon)$ for all $\hat{\omega}_0$ and hence that $K_1 \in D(\zeta\varepsilon)$. Thus it is common knowledge that $K_1 \in D(\zeta\varepsilon)$. Iterating the argument it follows that $K_1 \in \bigcap_{n=1}^{\infty} D(\zeta^n\varepsilon) = \{\bar{K}\}$. In contrast if (26) fails then for some beliefs compatible with CK1, the aggregate capital stock at $t = 1$ implied by the agents' optimal plans fails to satisfy $K_1 \in D(\varepsilon)$.

It follows that it is common knowledge that $K_1 = \bar{K}$ if and only if (26) holds. For agents at time $t = 1$ the situation is identical to the situation at time $t = 0$. Thus at

$t = 1$ agents will conclude that $K_2 = \bar{K}$ and this implies that it will be the case that $K_2 = \bar{K}$ and this will also be common knowledge for agents at $t = 0$. By induction it follows that the fact that the equilibrium path will be $K_t = \bar{K}$, for all t, is common knowledge. ■

We remark that this stability result is local and can provide a refinement criterion in the case of multiple steady states.

Note the crucial role played by the short planning horizon in going from weak to strong eductive stability. Because agents at time $t = 0$ care only about outcomes at $t = 0, 1$, the weak eductive stability result $K_1 \in D(\zeta\varepsilon)$, can be iterated to show common knowledge that $K_1 = \bar{K}$, and hence common knowledge that the next generation of agents will be in an identical position.

As an exercise for illustrating our results, we specify particular functional forms and conduct numerical analysis. Assume utility is time separable and takes the constant relative risk-aversion form

$$u(c_1, c_2) = \frac{1}{1-\sigma}\left(c_1^{1-\sigma} + c_2^{1-\sigma} - 2\right),$$

for $\sigma > 0$, and assume that production is Cobb-Douglas, $f(K, L) = K^\theta L^{1-\theta}$. In this case there is a unique positive steady-state level of capital, and parameter values for θ and σ completely characterize the model. For all parameters examined – $\theta \in (0, 1)$ and $\sigma \in (0, 100)$ – the steady state is strongly eductively stable.

This example provides a striking contrast to the coordination problems we have demonstrated for the RBC model with infinitely-lived agents.

References

[1] Benhabib, J. and C. Bull (1988), Conjectural equilibrium bounds, in Nonlinear Dynamics and Economics and Social Sciences, ed. M. Galeotti, L. Geronazzo and F. Gori. Pitagori Editrice Bologna.

[2] Benhabib, J. and R.A. Farmer (1994), Indeterminacy and increasing returns, Journal of Economic Theory 63, 19-41.

[3] Bernheim, D. (1984), Rationalizable strategic behavior, Econometrica 52, 1007-1028.

[4] Branch, W.A. and B. McGough, Business cycle amplification with heterogeneous expectations, Economic Theory 47, 395-421.

[5] Bray, M. and N. Savin (1986), Rational expectations, equilibria, learning, and model specification, Econometrica 54, 1129-1160.

[6] Brock, W.A. and C. Hommes (1997), A rational route to randomness, Econometrica 65, 1059–1095.

[7] Chiappori, P.-A. and R. Guesnerie (1991), Sunspot equilibria in sequential markets models. In Handbook of Mathematical Economics, ed. W. Hildenbrand and H. Sonnenschein. Amsterdam: North-Holland, 1684-1760.

[8] Desgranges, G. and S. Ghosal (2010), P-stable equilibrium: definition and some properties, Warwick economic research paper No. 952.

[9] Diamond, P.A. (1965), National debt in a neoclassical growth model, American Economic Review 55, 1126-1150.

[10] Eusepi, S. and B. Preston (2011), Expectations, learning and business cycle fluctuations, American Economic Review 101, 2844-2872.

[11] Evans, G.W. and R. Guesnerie (1993), Rationalizability, strong rationality and expectational stability, Games and Economic Behavior 5, 632-646.

[12] Evans, G.W. and R. Guesnerie (2005), Coordination on saddle-path solutions: the eductive viewpoint – linear multivariate models, Journal of Economic Theory 124, 202-229.

[13] Evans, G.W. and S. Honkapohja (1995), Local convergence of recursive learning to steady states and cycles in stochastic nonlinear models," Econometrica 63, 195-206.

[14] Evans, G.W. and S. Honkapohja (2001), Learning and Expectations in Macroeconomics. Princeton University Press, Princeton, New Jersey.

[15] Evans, G.W. and S. Honkapohja (2003), Expectations and the stability problem for optimal monetary policies, Review of Economic Studies 70, 807–824.

[16] Evans, G.W. and S. Honkapohja (2009), Learning and macroeconomics, Annual Review of Economics 1, 421-521.

[17] Evans, G.W., S. Honkapohja and K. Mitra (2009), Anticipated fiscal policy and adaptive learning, Journal of Monetary Economics 56, 930-953.

[18] Evans, G.W. and B. McGough (2012), Learning to optimize, mimeo.

[19] Gauthier, S. and R. Guesnerie (2004), Comparing expectational stability criteria in dynamical models: a preparatory overview, Delta DP 2004-9.

[20] Guesnerie, R. (1992), An exploration of the eductive justifications of the rational-expectations hypothesis, American Economic Review, 82, 1254-1278.

[21] Guesnerie, R. (2002), Anchoring economic predictions in common knowledge, Econometrica 70, 439-480.

[22] Guesnerie, R. (2005), Assessing Rational Expectations 2: "eductive" stability in economics. Cambridge MA: MIT Press.

[23] Guesnerie, R. and P. Jara-Moroni (2011), Expectational coordination in simple economic contexts: concepts and analysis with emphasis on strategic substitutabilities, Economic Theory 47, 205-246.

[24] Guesnerie, R. and M. Woodford (1991), Stability of cycles with adaptive learning rules, in Equilibrium Theory and Applications, Proceedings of the Sixth International Symposium in Economic Theory and Econometrics, ed. W. Barnett et al., Cambridge University Press, Cambridge UK, 111-134.

[25] Hommes, C. H. (2011), The heterogeneous expectations hypothesis: some evidence from the lab, Journal of Economic Dynamics and Control 35, 1-24.

[26] Hommes, C. H. and F. Wagener (2010), Does eductive stability imply evolutive stability?, Journal of Economic Behavior and Organization 75, 25-39.

[27] Howitt, P. (1992), Interest rate control and nonconvergence to rational expectations, Journal of Political Economy 100, 776-800.

[28] Kurz, M. and M, Motolese (2001), Endogenous uncertainty and market volatility, Economic Theory 17, 497-544.

[29] LaSalle, J.P. (1986), The Stability and Control of Discrete Processes. New York: Springer-Verlag.

[30] Kydland, F.E. and E.C. Prescott (1982), Time to build and aggregate fluctuations, Econometrica 50, 1345-1370.

[31] Long, J.B. and C.I. Plosser (1983), Real business cycles, Journal of Political Economy 91, 39-69.

[32] Lucas, Jr., Robert E. (1978), Asset prices in an exchange economy, Econometrica 46, 1429-1445.

[33] Marcet, A. and T.J. Sargent (1989), Convergence of least squares learning mechanisms in self-referential linear stochastic models, Journal of Economic Theory 48, 337-368.

[34] Matsui, A. and D. Oyama (2006), Rationalizable foresight dynamics, Games and Economic Behavior 56, 299-322.

[35] Morris, S. and H.S. Shin (1998), Unique equilibrium in a model of self-fulfilling attacks, American Economic Review 88, 587-597.

[36] Pearce, D. (1984), Rationalizable strategic behavior and the problem of perfection, Econometrica, July 1984, 52, 1029-1050.

[37] Prescott, E.C. (1986), Theory ahead of business cycle measurement, Carnegie-Rochester Conference Series on Public Policy 25 (Autumn), 11-44.

[38] Preston, B (2006), Adaptive learning, forecast-based instrument rules and monetary policy, Journal of Monetary Economics 53, 507-535.

[39] Sargent, T.J. (1993), Bounded Rationality in Macroeconomics, Oxford University Press, Oxford UK.

[40] Woodford, M. (1990), Learning to believe in sunspots, Econometrica 58, 277-307.